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PAST A WEDGE

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ABSTRACT

The nonlinear stability of a compressible flow past a wedge is investigated in the hypersonic limit. The analysis follows the ideas of a weakly nonlinear approach as first detailed by Smith (1979). Interest is focussed on Tollmien–Schlichting waves governed by a triple deck structure and it is found that the attached shock can profoundly affect the stability characteristics of the flow. In particular, it is shown that nonlinearity tends to have a stabilising influence. The nonlinear evolution of the Tollmien–Schlichting mode is described in a number of asymptotic limits which were first identified by Cowley and Hall (1990) in their linearised account of the current problem.

§1. Introduction.

This investigation is an extension of the work of Cowley & Hall (1990) who were concerned with the linear stability of hypersonic flow around a wedge of small angle. Here we aim to examine the effects of including nonlinearity in the solutions for the viscous modes of instability considered in Cowley & Hall.

Recently, there has been a resurgence in research of hypersonic flows, motivated by the interest in developing high speed aircraft. In compressible flow work it is necessary to specify a functional relationship between the temperature and the viscosity of the fluid. Frequently Chapman's law is used in which the viscosity is taken to be directly proportional to the temperature. Although this law can be rather unrealistic for high speed flows, stability analysis using this relationship is often more straightforward than for any other commonly used law. Cowley & Hall (1990), hereafter referred to as CH, used Chapman's in their work and we shall follow their lead here. A much more accurate relation, Sutherland's law, is used by Blackaby, Cowley & Hall (1990) in their investigation of the instability of hypersonic flow past a flat plate. They obtained smaller growth rates for the vorticity mode under consideration compared to those when Chapman's law was assumed. In this instance we know that Chapman's law is a bad approximation for the vorticity mode since it is concentrated in the layer where there is substantial variation in the basic temperature. However, since we are going to be concerned with the stability of Tollmien–Schlichting waves which by their nature are dominated by viscous effects close to the wall it is be to hoped that the results obtained here using Chapman's law will not be vastly different to those found should a more accurate law be used.

At hypersonic speeds real gas effects will be important but we do not take them into account in this study. These have been considered by Fu, Hall & Blackaby (1990) in their investigation of the instability of Görtler vortices at hypersonic speeds, in which they also assumed Sutherland's law. The fluid was taken to be an ideal gas and the effects of gas dissociation on the stability of the modes considered were obtained. The Görtler instability at hypersonic speeds has also been investigated by Hall & Fu (1989) and Dando & Seddougui (1991).

As already mentioned, in the present study attention is focussed on (viscous) Tollmien-Schlichting waves. CH investigated the effect of an attached shock on the viscous and inviscid modes in the compressible boundary layer on a thin wedge. For a discussion of the inviscid modes the reader is referred to Mack (1987). Smith & Brown (1990)

have investigated inviscid modes at high Mach numbers and in the absence of a shock. These authors concentrated on the vorticity mode while CH, in their study of the inviscid modes, looked at the acoustic modes which differ from the vorticity modes in as much as they are not concentrated within a thin layer. Blackaby, Cowley & Hall (1990) discuss the relationship which exists between these various modes. For large Mach numbers the inviscid modes have larger growth rates than the viscous ones but nevertheless the effect of the shock on the viscous modes is of some importance. In fact CH show that the presence of the shock greatly affects the viscous modes.

The wavelengths and frequency of the viscous instability in a compressible boundary layer at hypersonic speeds were given by Smith (1989). These form the basis of the scalings adopted in the present analysis. CH described the interactive triple-deck structure governing disturbances to a hypersonic flow over a wedge. The basic flow is given by an exact steady solution of the governing equations for a constant density gas and we follow CH and assume for simplicity that the wedge is insulated. As mentioned by CH the analysis can be extended to consider an isothermal flow where heat transfer at the surface is allowed. CH were concerned with the stability of the basic flow to viscous modes and as a first analysis they sought conditions under which non-parallel effects could be neglected. They showed that in order to do this it is necessary to make the Newtonian approximation and this yields a lower bound on the Mach number. A discussion of when non-parallel effects become important for hypersonic flow is given by Smith (1989) and this gives the same lower bound for the Mach number. CH also selected the angle between the surface of the wedge and the shock so that the shock occurs in the upper deck of the triple-deck structure. In this way the influence of the shock wave on the Tollmien-Schlichting waves can be investigated. As in classical triple–deck analysis as the perturbation to the basic flow is increased in amplitude from an infinitesimal size to a point where nonlinearity must be accounted for, it is the equations governing the disturbance in the lower deck which are the first to become nonlinear. We choose to retain the nonlinearities in the lower deck equations since the aim of the present study is to investigate these effects. The above requirements give bounds on the position of the shock or alternatively bounds on the size of the local Mach number with respect to the size of the Reynolds number, which will be taken as large for the following analysis.

CH determined the linearised conditions which hold at the shock. Since the nonlinear effects are unimportant in the upper deck (where the shock occurs) to the order of approximation considered both by CH and the present study these linearised shock conditions are appropriate. It is found that the effect of the shock is weak.

The linear results of CH for the Tollmien–Schlichting waves will be presented in §4, however, we give a brief summary of their results here. CH determined the nonlinear system governing the instability of Tollmien–Schlichting waves over a thin wedge in the presence of a shock. They linearised this system in order to determine the dispersion relation relevant to small disturbances and neutral solutions were presented for various positions of the shock. Since the disturbances are three–dimensional there exists an infinity of solutions. CH presented asymptotic solutions for the neutral wavenumbers which agree well with their numerical results. In particular, they showed that as the position of the shock, $\bar{y}_s$, tends to infinity the shockless dispersion relation derived by Smith (1989) is recovered. CH also gave a discussion on the temporal growth of the linearised modes.

CH proceeded to investigate the effect of the shock on neutral, two-dimensional inviscid modes of instability having wavelengths scaled on the boundary layer thickness. These modes are concentrated in the temperature adjustment layer centred at the generalised inflection point and solutions were given for the acoustic modes alluded to earlier. (A discussion of the vorticity modes in the absence of a shock is given by Smith & Brown (1990).) However, CH show that the shock has a negligible effect on the acoustic modes unless the distance of the shock from the surface is of the order of the boundary layer thickness.

Before we give an outline of the paper we mention some additional recent investigations of instability of hypersonic flow. Balsa & Goldstein (1990) investigated the high Mach number instability of supersonic mixing layers using asymptotic theory. They use the WKB method and solve the compressible Rayleigh equation for a non-isothermal free shear flow, so in a sense their solutions are similar to CH. This work is also related to that on the vorticity mode by Smith & Brown (1990). An investigation of nonlinear effects on the acoustic modes at hypersonic speeds described by CH is given by Goldstein & Wundrow (1990). Many aspects of theoretical research into hypersonic flow are discussed in the recent paper by Brown et al. (1991).

The plan of the paper is as follows. In §2 we describe the basic flow situation for hypersonic flow around a thin wedge, and give the equations governing the flow. The

scalings are chosen so that we are able to neglect non-parallelism, to include the effects of nonlinearity in the lower deck of the triple-deck structure and to ensure that the shock is located in the upper deck. The triple-deck structure for hypersonic flow is described in §3 and the scalings chosen to eliminate some of the variables for convenience. In addition, we state the linear conditions which hold at the shock (and which are described in detail in CH). In §4 we consider the nonlinear solutions of the governing system of equations. Rather than attempt a full numerical solution of the nonlinear equations we execute a weakly nonlinear analysis of the system in a manner virtually identical to that of Smith (1979) who studied the nonlinear stability of an incompressible boundary layer over a flat plate. In this way we derive an amplitude equation of classical Stuart–Watson type to describe the evolution of the disturbance. Solutions of this evolution equation are discussed in §5 where we also draw some conclusions.

§2. The basic flow.

We consider the flow of a compressible fluid over a wedge of semi–angle θ. As in CH, we suppose that the supersonic flow has velocity magnitude $\hat{U}$ and we consider the situation when the wedge is symmetrically aligned with the oncoming flow. Then symmetrical shocks develop on either side of the wedge. We denote the semi–angle of the shocks by σ and so the angle between the shock and the wedge is $\phi = \sigma - \theta$. This notation is chosen to be consistent with CH and the situation is illustrated in Figure 1.

We take $(\hat{x}, \hat{y}, \hat{z})$ co–ordinates with $\hat{x}$ denoting the distance along the upper surface of the wedge, $\hat{y}$ the distance normal to the wedge face and $\hat{z}$ as the spanwise co–ordinate. We suppose that the corresponding velocities are $(\hat{u}, \hat{v}, \hat{w})$ and denote quantities upstream of the shock and in the region between the shock and wedge by the subscripts u and s respectively.

We assume that the fluid is a perfect gas so we have the equation of state

$$\hat{p} = (\gamma - 1)\frac{c_p}{\gamma}\hat{\rho}\hat{T}, \tag{2.1a}$$

where $\hat{p}$, $\hat{\rho}$ and $\hat{T}$ denote the pressure, density and temperature of the fluid respectively, whilst γ is the ratio of specific heats and c_p is the specific heat at constant pressure. Then the speed of sound a_u in the upstream flow is defined by

$$a_u^2 = \frac{\gamma\hat{p}_u}{\hat{\rho}_u} = (\gamma - 1)\hat{h}_u, \tag{2.1b}$$

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where $\hat{h}_u = c_p \hat{T}$ is the enthalpy of the fluid. It follows that the upstream Mach number is given by

$$M_u = \frac{\hat{U}}{a_u}. \tag{2.1c}$$

The inviscid solution for hypersonic flow over a wedge is well known (see for example Hayes & Probstein (1966)) for, if we define $\epsilon = \hat{\rho}_u/\hat{\rho}_s$, then we have

$$\epsilon = \left(\frac{\gamma - 1}{\gamma + 1} \right) \left(1 + \frac{2}{(\gamma - 1)M_u^2 \sin^2 \sigma} \right), \tag{2.2a}$$

$$\frac{\hat{p}_s}{\hat{p}_u} = 1 + \gamma M_u^2 (1 - \epsilon) \sin^2 \sigma, \tag{2.2b}$$

$$\frac{\hat{h}_s}{\hat{h}_u} = 1 + \frac{1}{2}(\gamma - 1)(1 - \epsilon^2)M_u^2 \sin^2 \sigma, \tag{2.2c}$$

$$\tan \phi = \epsilon \tan \sigma, \tag{2.2d}$$

$$(\hat{u}_\parallel)_u = (\hat{u}_\parallel)_s = \hat{U} \cos \sigma, \qquad (\hat{u}_\perp)_u = -\hat{U} \sin \sigma, \qquad (\hat{u}_\perp)_s = -\epsilon \hat{U} \sin \sigma, \tag{2.2e}$$

where $\hat{u}_\parallel$ and $\hat{u}_\perp$ are the velocity components parallel and perpendicular to the shock. Thus we see from (2.2e) that the magnitude of the fluid velocity in the region between the shock and the wedge is given by

$$\hat{U}_s = \hat{U} \cos \sigma \left(1 + \epsilon^2 \tan^2 \sigma \right)^{\frac{1}{2}}, \tag{2.3}$$

which yields the slip velocity along the wedge. Thence, from (2.1c), (2.2c) and (2.3) the Mach number in the shock layer is given by

$$M_s^2 = \frac{2M_u^2 \cos^2 \sigma \left(1 + \epsilon^2 \tan^2 \sigma \right)}{2 + (\gamma - 1)(1 - \epsilon^2)M_u^2 \sin^2 \sigma}, \tag{2.4a}$$

whence

$$M_u^2 = \frac{2M_s^2}{2 \cos^2 \sigma \left(1 + \epsilon^2 \tan^2 \sigma \right) - (\gamma - 1)(1 - \epsilon^2)M_s^2 \sin^2 \sigma}. \tag{2.4b}$$

Since we are interested in Tollmien–Schlichting instabilities of this basic flow we need to consider the governing equations which comprise the continuity, Navier–Stokes and energy equations for a viscous, compressible fluid. We non–dimensionalise these equations using the values of the flow quantities between the shock and the wedge. We write

$$(\hat{x}, \hat{y}, \hat{z}) = L(x, y, z), \qquad (\hat{u}, \hat{v}, \hat{w}) = \hat{U}_s(u, v, w),$$

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where the lengthscale L is the distance from the tip of the wedge to the position of interest. We non–dimensionalise time with respect to $L/\hat{U}_s$, pressure with respect to $\hat{\rho}_s \hat{U}_s^2$ and the other variables with respect to their values in the shock layer. If we define a Reynolds number for the flow by

$$Re = \frac{\hat{\rho}_s \hat{U}_s L}{\hat{\mu}_s},\tag{2.5}$$

where $\hat{\mu}_s$ is a typical viscosity then the flow is governed by the equations

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0,\tag{2.6a}$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \frac{1}{Re}\left[2\nabla.(\mu \mathbf{e}) + \nabla\left((\mu' - \frac{2}{3}\mu)\nabla.\mathbf{u}\right)\right],\tag{2.6b}$$

and

$$\rho \frac{DT}{dt} = (\gamma - 1)M_s^2 \frac{Dp}{Dt} + \frac{1}{PrRe}\nabla.(\mu\nabla T) + \frac{(\gamma - 1)M_s^2}{Re}\Phi,\tag{2.6c}$$

where ρ, p and T are the non–dimensional density, pressure and temperature. The shear and bulk viscosities μ and μ' have both been non–dimensionalised with respect to $\hat{\mu}_s$ and Pr is the Prandtl number. Finally, the components of the rate of strain tensor are

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

and the dissipation function Φ is given by

$$\Phi = 2\mu \mathbf{e} : \mathbf{e} + \left(\mu' - \frac{2\mu}{3}\right)(\nabla.\mathbf{u})^2.$$

In the following analysis we shall assume that the Reynolds number is large. From (2.1a) the non–dimensional equation of state is

$$\gamma M_s^2 p = \rho T,\tag{2.7a}$$

and the non–dimensional enthalpy of the flow is given by

$$h = T.\tag{2.7b}$$

As in CH we shall restrict attention to the case when there is no heat transfer on the wedge surface so that $\partial T/\partial y = 0$ when $y = 0$ and the Prandtl is taken to be unity. Then the temperature of the fluid at the wall is given by

$$T_w = 1 + \frac{1}{2}(\gamma - 1)M^2,\tag{2.8}$$

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where the subscript s has been dropped from the Mach number.

The analysis presented in this paper is based on the assumption that the temperature dependence of the viscosity is described by Chapman's law; i.e.

$$\mu = CT, \tag{2.9}$$

where C is a constant. The argument leading to the choice of scalings shown below was given by CH for the more realistic Sutherland's law as well as for Chapman's law. However, since the results presented in CH are restricted to the Chapman law scenario and since our attention is also focussed on this case we shall just give the scalings appropriate to this situation here.

Then, following CH, we introduce the scalings

$$\sigma = \Sigma M^{\frac{7}{8}} Re^{-\frac{3}{16}} \qquad \text{and} \qquad (\gamma - 1)M^2\sigma^2 = \Gamma\Sigma^2, \tag{2.10a, b}$$

where Σ and Γ are $O(1)$ constants. If (2.10a) is satisfied for σ small then the shock will occur in the upper deck of the triple–deck structure and the effect of the the shock on the growth rate of the Tollmien–Schlichting waves can easily be determined. Consequently, we are assuming that ϕ, the angle between the shock and the wedge face, is small.

The problem is greatly simplified if the contributions from non-parallel effects can be neglected in a rational manner. This will be the case if the disturbance develops over a small streamwise distance. It can be deduced from the basic flow solution (2.2a) together with (2.4a) that $\epsilon << 1$ and for $M_u^2 > 0$ in (2.4b) we are forced to make the 'Newtonian' assumption

$$(\gamma - 1) << 1, \tag{2.11}$$

if non-parallel effects are to be ignored. The scalings (2.10) take account of these conditions and these considerations furnish a lower bound for σ. An upper bound for σ is obtained by including the effects of nonlinearity in the lower deck which is the aim of the present study. In CH the linearised problem was obtained from the nonlinear problem they initially formulated. As in CH we intend to apply linearised shock conditions in the upper deck and this consideration whilst simultaneously demanding nonlinearity in the lower deck yields the desired upper bound on σ. Specifically, the neglect of the non-parallelism and the imposition of nonlinearity imposes the constraints

$$Re^{-\frac{1}{10}} << \sigma << Re^{-\frac{1}{38}}, \tag{2.12a}$$

and so, from (2.10a)

$$Re^{\frac{1}{10}} << M << Re^{\frac{7}{38}}.$$ (2.12b)

For a detailed description of the argument leading to (2.10) and (2.12) the reader is once more referred to CH.

If the upper bound on σ is satisfied then, as discussed in CH, we are able to ignore viscous effects in the entropy and shear waves which are inevitably produced when the pressure/acoustic wave meets the shock. With the scales described above and assuming that the shock has a viscous internal layer the entropy and shear waves will have $y-$ scales much less than the width of the upper deck.

From (2.10b) if T_w is given by (2.8) then

$$T_w \sim \frac{1}{2}\Gamma\Sigma^2\sigma^{-2} >> 1,$$ (2.13a)

and to satisfy $M_u^2 > 0$ in (2.4b) we require

$$\Gamma\Sigma^2 < 2.$$ (2.13b)

The position of the shock is given by (2.2d). Thus, in cartesian co-ordinates at the shock we have, from (2.2a) and (2.10a) that

$$y = x\epsilon\tan\sigma \approx Re^{\frac{3}{16}}M^{-\frac{23}{8}}\Sigma^{-1}x.$$ (2.14)

We re-emphasize the point here that much of the above is described in much greater detail by CH.

§3. The triple–deck structure.

It was shown by CH that the hypersonic flow described in the previous section is governed by a triple–deck structure. In deriving the relevant scales within each zone it is convenient to scale out a number of the variables. The triple deck structure relating to a hypersonic flow situation of the type we have here was deduced by Smith (1989) in the course of his investigations into the stability of supersonic flows. It is possible to use Smith's results to infer the expansions which allow μ_w, T_w and M to be eliminated from the governing equations. These were given by CH but will be repeated here for convenience.

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The $x-$, $z-$ and $t-$ scalings will be the same in each deck and are given by

$$x = 1 + Re^{-\frac{3}{8}} C^{\frac{3}{8}} T_w^{\frac{3}{2}} M^{\frac{3}{4}} \lambda^{-\frac{5}{4}} X, \qquad (3.1a)$$

$$z = Re^{-\frac{3}{8}} C^{\frac{3}{8}} T_w^{\frac{3}{2}} M^{-\frac{1}{4}} Z, \qquad (3.1b)$$

and

$$t = Re^{-\frac{1}{4}} C^{\frac{1}{4}} T_w M^{\frac{1}{2}} \lambda^{-\frac{3}{2}} \tau, \qquad (3.1c)$$

where $C = \mu_w/T_w$ is the Chapman constant and $\lambda = \hat{\lambda} x^{-\frac{1}{2}}$, with $\hat{\lambda} = 0.332$, is the Blasius boundary layer skin–friction from the undisturbed middle deck solution.

§3.1 The lower deck

The lower deck is the region in which the nonlinearity of the problem manifests itself and the equations here necessarily contain viscous terms since it is this layer which is required to reduce the disturbance velocities to zero at the wedge surface. Indeed the relevant equations in the remaining zones of the triple–deck are linear as we shall describe presently. Within the lower deck we have scalings which take the forms

$$y = Re^{-\frac{5}{8}} C^{\frac{5}{8}} T_w^{\frac{3}{2}} M^{\frac{1}{4}} \lambda^{-\frac{3}{4}} Y, \qquad (3.2a)$$

$$u \sim Re^{-\frac{1}{8}} C^{\frac{1}{8}} T_w^{\frac{1}{2}} M^{\frac{1}{4}} \lambda^{\frac{1}{4}} U, \qquad (3.2b)$$

$$v \sim Re^{-\frac{3}{8}} C^{\frac{3}{8}} T_w^{\frac{1}{2}} M^{-\frac{1}{4}} \lambda^{\frac{3}{4}} V, \qquad (3.2c)$$

$$w \sim Re^{-\frac{1}{8}} C^{\frac{1}{8}} T_w^{\frac{1}{2}} M^{-\frac{3}{4}} \lambda^{\frac{1}{4}} W, \qquad (3.2d)$$

$$p \sim \gamma^{-1} M^{-2} + Re^{-\frac{1}{4}} C^{\frac{1}{4}} M^{-\frac{3}{2}} \lambda^{\frac{1}{2}} P, \qquad (3.2e)$$

$$T \sim T_w, \qquad (3.2f)$$

$$\rho \sim T_w^{-1}. \qquad (3.2g)$$

We substitute these expressions into $(2.6a, b)$ which give at leading order

$$U_X + V_Y + W_Z = 0, \qquad (3.3a)$$

$$U_\tau + UU_X + VU_Y + WU_Z = U_{YY}, \qquad (3.3b)$$

$$P_Y = 0, \qquad (3.3c)$$

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$$W_\tau + UW_X + VW_Y + WW_Z = -P_Z + W_{YY}. \qquad (3.3d)$$

The boundary conditions pertinent to system (3.3) require no–slip conditions on the wedge face so that

$$U = V = W = 0 \qquad \text{at} \quad Y = 0, \qquad (3.4a)$$

and the solutions must match with those in the middle deck as $Y \longrightarrow \infty$ so that

$$U \longrightarrow Y + A(X, Z, \tau), \qquad W \longrightarrow 0, \qquad \text{as} \quad Y \longrightarrow \infty, \qquad (3.4b)$$

where A is a displacement function which is to be determined.

§3.2 The middle deck

The middle deck covers the extent of the undisturbed boundary layer. Since, from (2.13a), $T_w \gg 1$ we require a thin transition region to reduce T to its free–stream value of unity. This transition region has thickness $O\left((2\log T_w)^{-\frac{1}{2}}\right)$ in terms of the Dorodnitsyn–Howarth variable. Thus, the middle deck is itself composed of three regions: *(i)* a boundary layer region where $T \gg 1$, *(ii)* the transition region and *(iii)* the region above the transition region where $T \sim 1$.

The thickness of region *(i)* is $O\left(Re^{-\frac{1}{2}}C^{\frac{1}{2}}T_w\right)$ and if we define

$$y = Re^{-\frac{1}{2}}C^{\frac{1}{2}}T_w y^*, \qquad (3.5)$$

then to match with the lower deck solutions as $y^* \longrightarrow 0$ we find that the solutions take the forms

$$u \sim u_0^*(y^*) + Re^{-\frac{1}{8}}C^{\frac{1}{8}}T_w^{\frac{1}{2}}M^{\frac{1}{4}}\lambda^{-\frac{3}{4}}Au_{0y^*}^*, \qquad (3.6a)$$

$$v \sim -Re^{-\frac{1}{4}}C^{\frac{1}{4}}M^{-\frac{1}{2}}\lambda^{\frac{1}{2}}A_X u_0^*, \qquad (3.6b)$$

$$w \sim Re^{-\frac{1}{4}}C^{\frac{1}{4}}M^{-\frac{1}{2}}\lambda^{\frac{1}{2}}\left(R_0^* u_0^*\right)^{-1}D(X, Z, \tau), \qquad (3.6c)$$

$$p \sim \gamma^{-1}M^{-2} + Re^{-\frac{1}{4}}C^{\frac{1}{4}}M^{-\frac{3}{2}}\lambda^{\frac{1}{2}}P, \qquad (3.6d)$$

$$\rho \sim R_0^*(y^*) + Re^{-\frac{1}{8}}C^{\frac{1}{8}}T_w^{\frac{1}{2}}M^{\frac{1}{4}}\lambda^{-\frac{3}{4}}AR_{0y^*}^*. \qquad (3.6e)$$

Here $u_0^*(y^*)$ and $R_0^*(y^*)$ are the undisturbed velocity and density profiles respectively and $D_X = -P_Z$. For Chapman's law u_0^* takes the form of a Blasius function for a compressible flow.

In region *(ii)* we write

$$y = y_c - Re^{-\frac{1}{2}} C^{\frac{1}{2}} T_w (2 \log T_w)^{-\frac{1}{2}} Y^*, \tag{3.7}$$

where y_c is a constant which determines the position of the transition layer. Here the solutions are very similar to (3.6) with the T_w scales slightly altered for u and ρ. In particular, for u, $T_w^{\frac{1}{2}}$ is replaced by $T_w^{-\frac{1}{2}} (2 \log T_w)^{\frac{1}{2}}$. The solutions in region *(iii)* are similar to those in region *(i)* with the simplification that $u_0^* = R_0^* = 1$ here.

§3.3 The upper deck

Finally, in the upper deck the basic flow quantities take their free–stream values. Thus, as in CH, the pressure/acoustic waves have scalings given by

$$y = Re^{-\frac{3}{8}} C^{\frac{3}{8}} T_w^{\frac{3}{2}} M^{-\frac{1}{4}} \lambda^{-\frac{5}{4}} \bar{y}, \tag{3.8a}$$

$$u \sim 1 + Re^{-\frac{1}{4}} C^{\frac{1}{4}} M^{-\frac{3}{2}} \lambda^{\frac{1}{2}} \bar{u}, \tag{3.8b}$$

$$(v, w) \sim Re^{-\frac{1}{4}} C^{\frac{1}{4}} M^{-\frac{1}{2}} \lambda^{\frac{1}{2}} (\bar{v}, \bar{w}), \tag{3.8c,d}$$

$$p \sim \gamma^{-1} M^{-2} + Re^{-\frac{1}{4}} C^{\frac{1}{4}} M^{-\frac{3}{2}} \lambda^{\frac{1}{2}} \bar{p}, \tag{3.8e}$$

$$\rho \sim 1 + Re^{-\frac{1}{4}} C^{\frac{1}{4}} M^{\frac{1}{2}} \lambda^{\frac{1}{2}} \bar{\rho}, \tag{3.8f}$$

$$T \sim 1 + (\gamma - 1) Re^{-\frac{1}{4}} C^{\frac{1}{4}} M^{\frac{1}{2}} \lambda^{\frac{1}{2}} \bar{T}. \tag{3.8g}$$

If we substitute (3.8) into (2.6) and (2.7) and rearrange the resulting equations we find that the pressure $\bar{p}$ satisfies

$$\bar{p}_{XX} - \bar{p}_{\bar{y}\bar{y}} - \bar{p}_{ZZ} = 0. \tag{3.9}$$

The solutions in the upper deck must match with those in the middle deck as $\bar{y} \longrightarrow 0$. From the y–momentum equation we obtain the boundary condition that

$$\bar{p}_{\bar{y}} = A_{XX} \quad \text{at} \quad \bar{y} = 0, \tag{3.10}$$

while $\bar{p} = P$ when $\bar{y} = 0$. The remaining boundary condition is that

$$\bar{p} = 0 \quad \text{at} \quad \bar{y} = \bar{y}_s, \tag{3.11}$$

where $\bar{y}_s$ is the position of the shock. The analysis leading to this condition with our basic state is given in the Appendix of CH and will not be repeated here. The reader is

also referred to Moore (1954), Ribner (1954) and McKenzie & Westphal (1968) for the conditions which hold at the shock. This condition is obtained from the linear inviscid problem and will continue to hold for the nonlinear analysis presented in this paper since the nonlinearity is confined to the lower deck to the orders considered.

For Chapman's law the position of the shock is given by (2.14). Thus, from (3.8a), (2.13a) and (2.10a) we obtain

$$\bar{y}_s = \left(\frac{2}{\Gamma}\right)^{\frac{3}{2}} \frac{\lambda^{\frac{5}{4}}}{C^{\frac{3}{8}} \Sigma}.$$

(3.12)

We shall now follow the convention of referring to the solutions of (3.9) as acoustic waves.

§4. The nonlinear solution.

In general the solution of equations (3.3), (3.4) and (3.9–11) is a fully numerical task but it is possible to make analytical progress by implementing a weakly nonlinear analysis of the system. The method used here is identical in spirit to that proposed by Smith (1979) who investigated the nonlinear stability of an incompressible Blasius boundary layer to Tollmien–Schlichting type disturbances. We differ slightly from Smith (1979) by considering a three–dimensional mode as in CH but this poses no formal difficulties whatsoever.

Our objective is to monitor the streamwise development of a finite amplitude TS wave whose leading order wavenumbers in the streamwise and spanwise directions are α and β respectively and whose frequency is Ω. Consequently, we seek modes whose fundamental component is proportional to

$$E \equiv \exp\left[i\left(\alpha X + \beta Z - \int^{\tau} \Omega(q)dq\right)\right].$$

(4.1)

If the weakly nonlinear disturbance is allowed to develop in the vicinity of the linear neutral point and if the relative amplitude of the disturbance in the lower deck is $O(h)$, $h \ll 1$, then following Smith (1979) it is found that the amplitude A of the mode will evolve on a $O(h^2)$ lengthscale. However, there is a lower bound on the possible size of h which is derived by consideration of the non-parallelism of the flow. Using an argument formally identical to that of Hall & Smith (1984) it is straightforward to demonstrate that the following account is valid for all h in the range $O(Re^{-\frac{3}{32}} M^{\frac{3}{16}} T_w^{\frac{3}{8}}) << h << 1$. If the lower bound of this inequality is attained then the lengthscale over which the amplitude of the

perturbation modulates is identical to that over which the non-parallelism of the basic flow occurs. For full details of this aspect the reader is referred to Hall & Smith (1984).

From (3.1a) we observe that the linear neutral stability point has been non-dimensionalised so as to be at $x = 1$ and so we need to consider perturbations at the point

$$x = 1 + h^2 x_2. \tag{4.2a}$$

Since the skin friction λ is a function of x it too will be slightly perturbed from its neutral value λ_1 according to

$$\lambda = \lambda_1(1 + h^2 \lambda_2). \tag{4.2b}$$

Additionally, we choose to fix the frequency but allow the spanwise wavenumber to vary and write

$$\beta = \beta_1 + h^2 \beta_2. \tag{4.2c}$$

These perturbations imply that for $h << 1$ we seek solutions of the lower deck equations (3.3–4) and the upper deck problem (3.9–11) in the forms

$$(U, V, W, P, A, \bar{p}) = \left((1 + h^2 \lambda_2)Y, \mathbf{0}\right) + \sum_{n=1}^{3} h^n (U_n, V_n, W_n, P_n, A_n, \bar{p}_n) + O(h^4). \tag{4.3}$$

Finally, the boundary condition (3.4b) which describes the matching with the middle deck solutions now assumes the form

$$U \longrightarrow (1 + h^2 \lambda_2)(Y + A), \qquad W \longrightarrow 0 \quad \text{as} \quad Y \longrightarrow \infty. \tag{4.4}$$

To account for the slow modulation of the amplitude on streamwise lengthscales we introduce the co-ordinate $\tilde{X} = h^2 X$ and then by use of multiple scales we replace all X derivatives throughout according to the recipe

$$\frac{\partial}{\partial X} \longrightarrow \frac{\partial}{\partial X} + h^2 \frac{\partial}{\partial \tilde{X}}. \tag{4.5}$$

Substitution of (4.3–5) in (3.3–4) leads to a hierarchy of problems at increasing orders in h. We address these problems in turn.

§4.1 The first order problem

The solution for the $O(h)$ terms in (4.3) has the linear stability form

$$(U_1, V_1, W_1, P_1, A_1, \bar{p}_1) = \left(\hat{U}_1(\tilde{X}, Y), \hat{V}_1(\tilde{X}, Y), \hat{W}_1(\tilde{X}, Y), \hat{P}_1(\tilde{X}), \hat{A}_1(\tilde{X}), \hat{p}_1(\tilde{X}, \bar{y}) \right) E + c.c.,$$
(4.6)

where we recall that E is defined by (4.1) and here, as in the remainder of the paper, *c.c.* denotes complex conjugate.

The problem for the unknowns in (4.6) is precisely the linear problem considered by CH. It is convenient to define

$$\xi = (i\alpha)^{\frac{1}{3}} \left(Y - \frac{\Omega}{\alpha} \right) \qquad \text{and} \qquad \xi_0 = -\frac{i^{\frac{1}{3}}\Omega}{\alpha^{\frac{2}{3}}},$$
(4.7)

and then we obtain

$$\alpha \hat{U}_1 + \beta_1 \hat{W}_1 = \frac{i\beta_1^2 \hat{P}_1}{(i\alpha)^{\frac{2}{3}} Ai'(\xi_0)} \int_{\xi_0}^{\xi} Ai(q) dq,$$
(4.8)

where $Ai(\xi)$ denotes the Airy function. In order to satisfy the boundary conditions as $\xi \longrightarrow \infty$ then

$$\alpha^{\frac{5}{3}} Ai'(\xi_0) \hat{A}_1 = i^{\frac{1}{3}} \beta_1^2 \kappa \hat{P}_1,$$
(4.9)

with $\kappa \equiv \int_{\xi_0}^{\infty} Ai(q) dq$. From the continuity equation (3.3a) we deduce that

$$\hat{V}_1 = \frac{(i\alpha)^{\frac{2}{3}} \hat{A}_1}{\kappa} \left[Ai'(\xi) - Ai'(\xi_0) - \xi \int_{\xi_0}^{\xi} Ai(q) dq \right].$$
(4.10)

Consideration of the upper deck problem (3.9–11) yields the solution

$$\hat{p}_1 = \frac{\alpha^2}{\sqrt{\beta_1^2 - \alpha^2}} \frac{\sinh \left[(\beta_1^2 - \alpha^2)^{\frac{1}{2}} (\bar{y}_s - \bar{y}) \right]}{\cosh \left[(\beta_1^2 - \alpha^2)^{\frac{1}{2}} \bar{y}_s \right]} \hat{A}_1, \qquad \text{for} \quad \beta_1^2 > \alpha^2,$$
(4.11a)

and

$$\hat{p}_1 = \frac{\alpha^2}{\sqrt{\alpha^2 - \beta_1^2}} \frac{\sin \left[(\alpha^2 - \beta_1^2)^{\frac{1}{2}} (\bar{y}_s - \bar{y}) \right]}{\cos \left[(\alpha^2 - \beta_1^2)^{\frac{1}{2}} \bar{y}_s \right]} \hat{A}_1, \qquad \text{for} \quad \beta_1^2 < \alpha^2.$$
(4.11b)

Note that the solution (4.11b) for $\hat{p}_1$ does not decay as $\bar{y}_s \longrightarrow \infty$. Since $\hat{p}_1 = \hat{P}_1$ at $\bar{y} = 0$ we obtain from (4.9) and (4.11) the dispersion relation

$$\frac{(i\alpha)^{\frac{1}{3}} \beta_1^2 \kappa}{Ai'(\xi_0)} = \frac{(\beta_1^2 - \alpha^2)^{\frac{1}{2}}}{\tanh \left((\beta_1^2 - \alpha^2)^{\frac{1}{2}} \bar{y}_s \right)} \qquad \text{for} \quad \beta_1^2 > \alpha^2,$$
(4.12a)

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$$\frac{(i\alpha)^{\frac{1}{3}}\beta_1^2\kappa}{Ai'(\xi_0)} = \frac{(\alpha^2 - \beta_1^2)^{\frac{1}{2}}}{\tan\left((\alpha^2 - \beta_1^2)^{\frac{1}{2}}\bar{y}_s\right)} \qquad \text{for} \quad \beta_1^2 < \alpha^2. \tag{4.12b}$$

As mentioned by CH in the limit $\bar{y}_s \longrightarrow \infty$ for $\beta_1^2 > \alpha^2$ the relation (4.12a) becomes that obtained by Smith (1989) for hypersonic flow with no shock.

The neutral solution of (4.12b) for α and β_1 real is determined from the eigenrelation

$$(\beta_1^2 - \alpha^2)^{\frac{1}{2}} = c_2\alpha^{\frac{1}{3}}\beta_1^2\tanh\left((\beta_1^2 - \alpha^2)^{\frac{1}{2}}\bar{y}_s\right), \tag{4.13}$$

and $\xi_0 = -c_1 i^{\frac{1}{3}}$ where $c_1 \approx 2.3$ and $c_2 \approx 1.0$. The corresponding expression for $\beta_1^2 < \alpha^2$ is similar to (4.13) and CH obtained neutral solutions to these two equations for various values of $\bar{y}_s$. The dispersion relation admits an infinity of solutions and in Figure (2) we present some of these solutions for the three cases $\bar{y}_s = 1$, 4 and 16. For each $\bar{y}_s$ we observe that there is one mode for which $\alpha \longrightarrow 0$ as $\beta_1 \longrightarrow \infty$ and an infinity of modes all of which asymptote to $\alpha = \beta_1$ as $\beta_1 \longrightarrow \infty$. The Mach cone is defined by the line $\alpha = \beta_1$ so that above this line the acoustic waves are sinusoidal and supersonic whereas below this line the modes are subsonic. Hence for nearly all spanwise wavenumbers there are an infinity of supersonic modes and precisely one subsonic mode. We shall infer precise details concerning the asymptotic limit $\beta_1 \longrightarrow \infty$ in Section 5 where we shall also discuss solutions of the amplitude evolution equation for $\hat{A}_1(\tilde{X})$ which is to be derived.

§4.2. The second order problem.

By examining $O(h^2)$ terms in (3.3) and (3.9) we conclude that the solutions of these equations at this order take the forms

$$U_2 = U_{21}E^2 + U_{22} + U_{21}^{(c)}E^{-2}, \tag{4.14}$$

with similar expansions for V_2, W_2, P_2, A_2 and $\bar{p}_2$. Here the terms U_{21}, U_{22}, V_{21}, V_{22}, W_{21} and W_{22} are functions of $\tilde{X}$ and Y, $\bar{p}_{21}$ and $\bar{p}_{22}$ are functions of $\tilde{X}$ and $\bar{y}$ and the remaining terms are functions of $\tilde{X}$ alone.

From $O(h^2E^2)$ terms in (3.3) we find, using manipulations similar to those described by Smith (1979), that

$$\alpha U_{21} + \beta_1 W_{21} = \frac{i\alpha^2}{\kappa^2}\left(\hat{A}_1\right)^2(i\alpha)^{-\frac{2}{3}}\int_{\xi_0}^{\xi}\left[F(q) + Ai'(q)\int_{\xi_0}^{q}Ai(t)dt\right]dq + B_2(\tilde{X})\int_{\xi_0}^{\xi}Ai(2^{\frac{1}{3}}t)dt, \tag{4.15a}$$

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which, using the continuity equation (3.3a), leads to

$$V_{21} = -2i(i\alpha)^{-\frac{1}{3}} \int_{\xi_0}^{\xi} (\alpha U_{21} + \beta_1 W_{21}) \, d\xi. \tag{4.15b}$$

In (4.15) we have made the definitions

$$F(\xi) \equiv Ai(2^{\frac{1}{3}}\xi) \int_{2^{\frac{1}{3}}\xi_0}^{2^{\frac{1}{3}}\xi} \frac{dq}{[Ai(q)]^2} \int_{\infty}^{q} Ai(t)R(t)dt, \tag{4.16a}$$

where

$$R(2^{\frac{1}{3}}\xi) \equiv -2^{-\frac{2}{3}} \left[2Ai(\xi)Ai''(\xi) + Ai'(\xi_0)Ai'(\xi) \right]. \tag{4.16b}$$

Imposing the boundary condition (3.4b) as $\xi \longrightarrow \infty$ in (4.15a) gives

$$\lambda_1 \alpha A_{21} = \frac{i\alpha^2}{\kappa^2} \left(\hat{A}_1 \right)^2 (i\alpha)^{-\frac{2}{3}} \int_{\xi_0}^{\infty} \left[F + Ai'(\xi) \int_{\xi_0}^{\xi} Ai(t)dt \right] d\xi + B_2 \int_{\xi_0}^{\infty} Ai(2^{\frac{1}{3}}t)dt. \tag{4.17}$$

From $O(h^2 E^2)$ terms in the momentum equations $(3.3b, d)$ when evaluated at $Y = 0$ we obtain the relationship

$$(i\alpha)^{\frac{2}{3}} \left(\alpha \frac{\partial^2 U_{21}}{\partial \xi^2} + \beta_1 \frac{\partial^2 W_{21}}{\partial \xi^2} \right) \bigg|_{\xi=\xi_0} = 2i\beta_1^2 P_{21}, \tag{4.18}$$

and elimination of B_2 between (4.15a), (4.17) and (4.18) yields a first connection between P_{21} and A_{21}. A second relation between these quantities is obtained by studying the upper deck problem (3.9–11) where, on considering $O(h^2 E^2)$ terms we find that

$$\frac{\partial^2 \bar{p}_{21}}{\partial \bar{y}^2} - 4(\beta_1^2 - \alpha^2)\bar{p}_{21} = 0, \tag{4.19a}$$

with associated boundary conditions

$$\bar{p}_{21} = 0 \qquad \text{at} \quad \bar{y} = \bar{y}_s, \tag{4.19b}$$

$$\frac{\partial \bar{p}_{21}}{\partial \bar{y}} = -4\alpha^2 A_{21} \qquad \text{at} \quad \bar{y} = 0. \tag{4.19c}$$

Solution of (4.19), when combined with the matching requirement with the main deck solution that $\bar{p}_{21} = P_{21}$ at $\bar{y} = 0$ yields the relation

$$P_{21} = \frac{2\alpha^2 A_{21}}{(\beta_1^2 - \alpha^2)^{\frac{1}{2}}} \tanh \left(2(\beta_1^2 - \alpha^2)^{\frac{1}{2}} \bar{y}_s \right), \tag{4.20}$$

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for $\beta_1^2 > \alpha^2$ with an analogous expression when $\beta_1^2 < \alpha^2$. This second expression for P_{21}/A_{21} enables $B_2(\tilde{X})$ to be found, as previously described. The resulting expressions are complicated and so we shall not state them here.

Turning to the $O(h^2)$ mean flow terms $U_{22}, V_{22}, W_{22}, P_{22}, A_{22}$ and $\bar{p}_{22}$ in the governing equations we obtain the results that

$$V_{22} = 0, \tag{4.21a}$$

$$(i\alpha)^{\frac{2}{3}}\left(\alpha\frac{\partial^2 U_{22}}{\partial\xi^2} + \beta_1\frac{\partial^2 W_{22}}{\partial\xi^2}\right) = \left((i\alpha)^{\frac{1}{3}}\right)^{(c)}\hat{V}_1\left(\alpha\frac{\partial\hat{U}_1^{(c)}}{\partial\xi} + \beta_1\frac{\partial\hat{W}_1^{(c)}}{\partial\xi}\right) + c.c. \tag{4.21b}$$

This latter equation has solution

$$\alpha U_{22} + \beta_1 W_{22} = (i\alpha)^{-\frac{2}{3}}\alpha^2|\frac{\hat{A}_1}{\kappa}|^2\int_{\xi_0}^{\xi}dt\int_{\infty}^{t}f^{**}(q)dq, \tag{4.22a}$$

where

$$f^{**}(\xi) = i^{\frac{1}{3}}\left(Ai(\xi)\right)^{(c)}\left[Ai'(\xi) - Ai'(\xi_0) - \xi\int_{\xi_0}^{\xi}Ai(t)dt\right] + c.c. \tag{4.22b}$$

Applying the boundary condition (3.4b) as $\xi \longrightarrow \infty$ implies that

$$A_{22} = \frac{\alpha(i\alpha)^{-\frac{2}{3}}}{\lambda_1}|\frac{\hat{A}_1}{\kappa}|^2\int_{\xi_0}^{\infty}d\xi\int_{\infty}^{\xi}f^{**}(t)dt. \tag{4.23}$$

Finally, the upper deck problem (3.9-11) yields $\bar{p}_{22} = P_{22} = 0$.

§4.3 The third order problem.

At this order we determine the amplitude equation for the unknown function $\hat{A}_1(\tilde{X})$. This equation arises from a solvability condition on the $O(h^3E)$ terms in the systems (3.3) and (3.9-11), solutions of which are sought of the form

$$U_3 = EU_{31} + E^2U_{32} + E^3U_{33} + U_{34} + E^{-1}U_{31}^{(c)} + E^{-2}U_{32}^{(c)} + E^{-3}U_{33}^{(c)},$$

with similar expansions for V_3, W_3, A_3, P_3 and $\bar{p}_3$. The desired solvability criterion is obtained from consideration of the coefficients of $O(h^3E)$ terms in the lower deck equations (3.3) and this leads to

$$\frac{\partial^3}{\partial\xi^3}(\alpha U_{31} + \beta_1 W_{31}) - \xi\frac{\partial}{\partial\xi}(\alpha U_{31} + \beta_1 W_{31}) = -(i\alpha)^{-\frac{2}{3}}\frac{\partial G}{\partial\xi} + i\frac{\partial\hat{U}_1}{\partial\tilde{X}} - \beta_2\hat{W}_1, \tag{4.24a}$$

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where

$$
\begin{aligned}
-G(\xi, \tilde{X}) =& (i\alpha)^{-\frac{1}{3}}(\xi - \xi_0)\frac{\partial}{\partial \tilde{X}}(\alpha \hat{U}_1 + \beta_1 \hat{W}_1) + (i\alpha)^{\frac{2}{3}}(\xi - \xi_0)\lambda_2(\alpha \hat{U}_1 + \beta_1 \hat{W}_1) \\
&+ \alpha \lambda_2 \hat{V}_1 + i\beta_1 \beta_2 \hat{P}_1 + i(\alpha \hat{U}_1 + \beta_1 \hat{W}_1)^{(c)}(\alpha U_{21} + \beta_1 W_{21}) \\
&+ i(\alpha U_{22} + \beta_1 W_{22})(\alpha \hat{U}_1 + \beta_1 \hat{W}_1) + (i\alpha)^{\frac{1}{3}}\hat{V}_1 \frac{\partial}{\partial \xi}(\alpha U_{22} + \beta_1 W_{22}) \\
&+ (i\alpha)^{\frac{1}{3}}\hat{V}_1^{(c)} \frac{\partial}{\partial \xi}(\alpha U_{21} + \beta_1 W_{21}) + \left((i\alpha)^{\frac{1}{3}}\right)^{(c)} V_{21}\frac{\partial}{\partial \xi}(\alpha \hat{U}_1^{(c)} + \beta_1 \hat{W}_1^{(c)}).
\end{aligned}
$$

$$(4.24b)$$

The boundary conditions obtained from (3.4) are

$$U_{31}(\xi_0, \tilde{X}) = W_{31}(\xi_0, \tilde{X}) = 0, \tag{4.25a}$$

$$U_{31}(\infty, \tilde{X}) = A_{31}(\tilde{X}) + \lambda_2 \hat{A}_1(\tilde{X}), \qquad W_{31}(\infty, \tilde{X}) = 0. \tag{4.25b}$$

By evaluating the streamwise and spanwise momentum equations $(3.3b, d)$ at $\xi = \xi_0$ we get

$$i^{\frac{1}{3}}\alpha^{-\frac{2}{3}}\beta_1^2 P_{31} + i^{\frac{1}{3}}\alpha^{-\frac{2}{3}}\beta_1\beta_2\hat{P}_1 = \left(\alpha\frac{\partial^2 U_{31}}{\partial \xi^2} + \beta_1 \frac{\partial^2 W_{31}}{\partial \xi^2}\right)\bigg|_{\xi=\xi_0}. \tag{4.26}$$

The homogeneous forms of equation (4.24) and boundary conditions (4.25) are satisfied by $\alpha\hat{U}_1 + \beta_1\hat{W}_1$ and so (4.24–5) only has a solution for $\alpha U_{31} + \beta_1 W_{31}$ if a certain compatibility condition holds (as in Stuart (1960) and Watson (1960)). This condition is derived by considering the adjoint system of the problem as in Hall & Smith (1982). We multiply $(4.24a)$ by the function

$$K(\xi) \equiv Ai(\xi) - \frac{Ai'(\xi_0)}{\mathcal{L}'(\xi_0)}\mathcal{L}(\xi), \tag{4.27a}$$

where

$$\mathcal{L}(\xi) = Ai(\xi) \int_{\xi_0}^{\xi} \frac{dq}{[Ai(q)]^2} \left(\int_{\infty}^{q} Ai(s)ds\right), \tag{4.27b}$$

and integrate over the range (ξ_0, ∞). This gives

$$
-i^{\frac{1}{3}}\alpha^{-\frac{2}{3}}Ai(\xi_0)\beta_1^2 P_{31} - \frac{Ai'(\xi_0)}{\mathcal{L}'(\xi_0)}\alpha A_{31} =
$$

$$
\frac{Ai'(\xi_0)}{\mathcal{L}'(\xi_0)}\alpha\lambda_2\hat{A}_1 + (i\alpha)^{-\frac{2}{3}}\int_{\xi_0}^{\infty} K'(\xi)G(\xi, \tilde{X})d\xi + \int_{\xi_0}^{\infty} K(\xi)\left(i\frac{\partial \hat{U}_1}{\partial \tilde{X}} - \beta_2 \hat{W}_1\right)d\xi.
$$

$$(4.28)$$

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In the upper deck the problem for $\bar{p}_{31}$ takes the form (on equating terms of $O(h^3 E)$ in (3.9)),

$$\frac{\partial^2 \bar{p}_{31}}{\partial \bar{y}^2} - (\beta_1^2 - \alpha_2)\bar{p}_{31} = 2i\alpha \frac{\partial \hat{p}_1}{\partial \tilde{X}} + 2\beta_1 \beta_2 \hat{p}_1, \qquad (4.29a)$$

with boundary conditions

$$\bar{p}_{31} = 0 \qquad \text{at} \quad \bar{y} = \bar{y}_s, \qquad (4.29b)$$

$$\frac{\partial \bar{p}_{31}}{\partial \bar{y}} = -\alpha^2 A_{31} + 2i\alpha \frac{\partial \hat{A}_1}{\partial \tilde{X}} \qquad \text{and} \qquad \bar{p}_{31} = P_{31} \qquad \text{at} \quad \bar{y} = 0. \qquad (4.29c)$$

Solution of (4.29) yields a second expression for

$$i^{\frac{1}{3}} \alpha^{-\frac{2}{3}} Ai(\xi_0) \beta_1^2 P_{31} + \frac{Ai'(\xi_0)}{\mathcal{L}'(\xi_0)} \alpha A_{31}$$

in terms of the solutions derived when considering the first and second order problems in Sections 4.1 and 4.2. Then, after considerable manipulation, substitution of (4.24b) for $G(\xi, \tilde{X})$ in (4.28) enables the desired evolution equation to be obtained. Since the expressions occurring in this equation are long and complicated we have relegated these details to the Appendix. In the following section we analyse some solutions of the amplitude equation and comment upon some of the implications.

§5. Results and discussion.

By implementing the procedures described in the previous section we obtain a classical evolution equation for the amplitude $\hat{A}_1(\tilde{X})$ which takes the general form

$$\frac{d\hat{A}_1}{d\tilde{X}} = (a_1 \beta_2 + a_2 \lambda_2)\hat{A}_1 + a_3 \hat{A}_1 |\hat{A}_1|^2, \qquad (5.1)$$

where the definitions of the complex valued constants a_1, a_2 and a_3 are given in the Appendix. As is usual in this type of weakly nonlinear analysis interest is primarily centred upon the sign of the real part of a_3. It is easily deduced from (5.1) that

$$\frac{d}{d\tilde{X}} \left(|\hat{A}_1|^2 \right) = [2Re(a_2 \lambda_2 + a_1 \beta_2)] |\hat{A}_1|^2 + 2Re(a_3)|\hat{A}_1|^4, \qquad (5.2)$$

and thence if $Re(a_3) > 0$ nonlinear effects are destabilising and the equilibrium state with amplitude

$$\left[\frac{-Re(a_3)}{Re(a_1 \beta_2 + a_2 \lambda_2)} \right]^{\frac{1}{2}}, \qquad (5.3)$$

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is unstable. Conversely, for $Re(a_3) < 0$ nonlinearity tends to stabilise a linearly unstable mode. (Here $Re(\Phi)$ denotes the real part of the complex number Φ.)

We now illustrate solution characteristics for the values of $\bar{y}_s$ considered previously. Firstly, in Figure (3) we show the dependence of the real part of the coefficient a_1 as functions of β_1 and where α is then given by the solution of the eigenrelation (4.12). We see from these diagrams that the behaviour of a_1 is markedly different for the subsonic mode as opposed to the remaining modes. Specifically, for sufficiently large $\bar{y}_s$ the real part of a_1 appears to be negative for all β_1 and for all modes except the subsonic one. Figure (4) shows that the constant a_2 has the property that $Re(a_2) < 0$ for all values of $\bar{y}_s$ and β_1. In particular, the subsonic mode has the greatest absolute value for $Re(a_2)$ and the ratio between the size of this constant for the subsonic mode to that for the supersonic modes increases as the shock position $\bar{y}_s$ moves away from the wedge surface.

When the amplitude equation (5.1) is linearised we observe that the sign of $Re(a_2 \lambda_2 + a_1 \beta_2)$ determines whether small amplitude Tollmien–Schlichting waves are stable or unstable according to a usual linear stability analysis. As previously described, it is the sign of $Re(a_3)$ which governs the effect of the nonlinear term on the stability of near neutral modes. Crucially, Figure (5) demonstrates that for all $\bar{y}_s$ and β_1 we have $Re(a_3) < 0$ and hence for each mode the effect of nonlinearity is stabilising. The size of the nonlinear term decreases as $\bar{y}_s$ increases and for sufficiently large $\bar{y}_s$ the nonlinear term relating to the subsonic mode far exceeds in value those for the remaining supersonic cases.

Above we have provided a brief description of the results of the weakly nonlinear calculations for a selection of shock positions $\bar{y}_s$. We now turn to consider some analytical results which are derivable in certain important limits. CH identified three particular cases in which asymptotic progress is easily accomplished. Firstly, we consider the case of a weak three–dimensionality effect in the TS disturbance (4.1), i.e. $\beta_1 << 1$. Then the solution of eigenrelation (4.12) takes the form

$$\alpha \sim \frac{(n + \frac{1}{2})\pi}{\bar{y}_s} + \left(\frac{\bar{y}_s}{2(n + \frac{1}{2})\pi} - \frac{c_2}{\bar{y}_s} \left(\frac{\bar{y}_s}{(n + \frac{1}{2})\pi} \right)^{\frac{2}{3}} \right) \beta_1^2 + \ldots, \qquad (5.4a)$$

where $n = 1, 2, \ldots$ and where we recall that $c_2 \approx 1.0$. Then, using the functions defined in the Appendix it follows that at leading orders in β_1 the evolution equation for $\hat{A}_1$ becomes

$$\frac{d\hat{A}_1}{d\tilde{X}} = -\frac{\beta_1^2 c_2^2}{\bar{y}_s Ai(\xi_0)} \left(\frac{\bar{y}_s}{(n+\frac{1}{2})\pi}\right)^{\frac{7}{3}} \times \Bigg[$$

$$\left\{\lambda_2 \left(\frac{(n+\frac{1}{2})\pi}{\bar{y}_s}\right)^{\frac{5}{3}} L_2 + \frac{\beta_2}{\beta_1} \left(\left(\frac{(n+\frac{1}{2})\pi}{\bar{y}_s}\right)^{\frac{5}{3}} L_3 + \frac{\bar{y}_s}{c_2^2} \left(\frac{(n+\frac{1}{2})\pi}{\bar{y}_s}\right)^{\frac{4}{3}} L_4\right)\right\} \hat{A}_1$$

$$+ \left(\frac{(n+\frac{1}{2})\pi}{\bar{y}_s}\right)^{\frac{7}{3}} \frac{L_5}{L_7} \hat{A}_1 |\hat{A}_1|^2 \Bigg],$$

where the constants L_j are defined in the Appendix. On evaluation this amplitude equation becomes

$$\frac{d\hat{A}_1}{d\tilde{X}} =$$

$$\left(\frac{\lambda_2 \beta_1^2}{\bar{y}_s^{\frac{1}{3}}(n+\frac{1}{2})^{\frac{2}{3}}}(-0.378, 0.273) + \frac{\beta_1 \beta_2}{\bar{y}_s^{\frac{1}{3}}(n+\frac{1}{2})^{\frac{2}{3}}} \left[(-0.001, -0.932) + \frac{\bar{y}_s^{\frac{4}{3}}}{(n+\frac{1}{2})^{\frac{1}{3}}}(0, 0.318)\right]\right) \hat{A}_1$$

$$+ \frac{\beta_1^2}{\bar{y}_s}(-0.279, 7.03 \times 10^{-2})\hat{A}_1|\hat{A}_1|^2.$$

$$(5.4b)$$

We may deduce from this equation that for $\beta_1 << 1$ all the coefficients in this amplitude equation become small. Further, on considering the coefficient of the linear term we see that variations in the perturbed spanwise wavenumber β_2 have little effect on the linear growth rate of the TS wave. Finally, for $\beta_2 = 0$ we have the possibility of a finite amplitude stable mode of size

$$\hat{A}_e = \frac{1.16\bar{y}_s^{\frac{1}{3}}}{(n+\frac{1}{2})^{\frac{1}{3}}}\sqrt{-\lambda_2}.$$

A second case for investigation is that of supersonic modes with $\alpha >> 1$. From Figure (2) we observed that such neutral modes have $\beta_1 \overset{\approx}{\sim} \alpha$ and, more accurately, we find that

$$\alpha = \beta_1 + \left(\frac{n\pi}{\bar{y}_s}\right)^2 \frac{1}{2\beta_1} - \left(\frac{n\pi}{\bar{y}_s}\right)^4 \frac{1}{8\beta_1^3} + \frac{1}{c_2\bar{y}_s}\left(\frac{n\pi}{\bar{y}_s}\right)^2 \beta_1^{-\frac{10}{3}} + \ldots, \qquad (5.5a)$$

where $n = 1, 2, \ldots$. Thus for $\beta_1 >> 1$ the possible α values for neutrally stable linear TS waves are separated by $O(1/n)$. In this case the amplitude equation asymptotes to

$$-\frac{\beta_1^5 \bar{y}_s^3 Ai(\xi_0)}{(n\pi)^2}\frac{d\hat{A}_1}{d\tilde{X}} = \left[\lambda_2 \beta_1^{\frac{5}{3}} L_2 + \beta_2 \left(\frac{\beta_1^5 \bar{y}_s^3 L_4}{(n\pi)^2}\right)\right]\hat{A}_1 + \frac{\beta_1^{\frac{7}{3}}(c_2 L_5 + 4L_6)}{(c_2 L_7 + 4L_8)}\hat{A}_1|\hat{A}_1|^2. \quad (5.5b)$$

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Performing the necessary evaluations yields

$$\frac{d\hat{A}_1}{d\tilde{X}} \approx \left[(-8.016, 5.781)\frac{n^2\beta_1^{-\frac{10}{3}}}{\bar{y}_s^3}\lambda_2 + i\beta_2\right]\hat{A}_1 + (-6.058, -0.377)\frac{n^2\beta_1^{-\frac{8}{3}}}{\bar{y}_s^3}\hat{A}_1|\hat{A}_1|^2, \quad (5.5c)$$

and hence we can obtain supercritically stable modes of amplitude $\approx 1.15\beta_1^{-\frac{1}{3}}\sqrt{-\lambda_2}$. We note that in this $\alpha, \beta \gg 1$ limit the scaled equilibrium amplitude is small and is independent of the perturbed spanwise wavenumber β_2.

A final regime for asymptotic investigation is that of the solitary subsonic mode in Figures (2), i.e. those modes for which $\alpha \longrightarrow 0$ as $\beta_1 \longrightarrow \infty$. The solution of eigenrelation (4.12a) in this case assumes the form

$$\alpha \sim \frac{1}{(c_2\beta_1)^3} + \cdots, \quad (5.6a)$$

and then the amplitude equation (5.1) simplifies, at leading order in β_1^{-1} to

$$\frac{c_2L_1 - 2Ai(\xi_0)}{\beta_1^2 c_2^3}\frac{d\hat{A}_1}{d\tilde{X}} = \left[\lambda_2\left(\frac{L_2}{c_2^5\beta_1^5}\right) + \frac{\beta_2(c_2L_3 + L_4)}{c_2^6\beta_1^6}\right]\hat{A}_1 + \frac{(c_2L_5 + 2L_6)}{c_2^7\beta_1^7(c_2L_7 + 2L_8)}\hat{A}_1|\hat{A}_1|^2,$$

or,

$$\frac{d\hat{A}_1}{d\tilde{X}} = \left[(-0.376, 0.117)\lambda_2\beta_1^{-3} + (0.125, -0.373)\beta_2\beta_1^{-4}\right]\hat{A}_1 \qquad (5.6b)$$
$$+ (-0.321, -8.83 \times 10^{-2})\beta_1^{-5}\hat{A}_1|\hat{A}_1|^2.$$

This evolution equation admits a stable finite equilibrium solution in which $\hat{A}_1 = O(\beta_1)$.

We remark that the three asymptotic limits briefly alluded to above agree well with the numerical solutions sketched in Figures (3)–(5). It is straightforward to make a comparison of the relative likelihoods of the equilibrium modes in the $\beta_1 \gg 1$ case. We have seen that for $\beta_1 \gg 1$ the supersonic modes have amplitudes $O(\beta_1^{-\frac{1}{3}})$ whereas the corresponding amplitude for the subsonic mode is much larger, $O(\beta_1)$. Hence, in practice, at large spanwise wavenumbers β_1 the subsonic mode will likely dominate the other modes and is therefore much more likely to be observable.

To conclude, we have presented an account of some properties of classical weakly nonlinear stability theory as applied to an, admittedly idealised, hypersonic flow problem originally considered by CH. Our main conclusion is that for all the parameter regimes investigated near linear neutrally stable modes appear to be supercritically stable; i.e. the nonlinear effects tend to stabilise the slightly unstable linearised modes.

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There is much further work which is possible in relation to this problem. The most obvious is that of a full nonlinear solutions of the lower deck equations (3.3) together with boundary conditions (3.4); work which when combined with the relevant upper deck problem clearly needs to be numerical in character. A second course of action is that of considering viscosity/ temperature dependences other than that of Chapman's law. As mentioned earlier CH and Blackaby *et. al.* (1990) consider situations in which the more physically realistic Sutherland's law is used although CH did not calculate any results for this case. The effect of changing the viscosity law on our results would be of great interest for, of course, if it transpires that the properties of our solutions are quite sensitive to the actual laws invoked then this raises the question as to which to use in future studies.

One complete subject area which has been deliberately excluded in both CH and the present study is that as to the nature of any nonlinearities which might occur in the vicinity of the shock layer position $\bar{y}_s$. The scalings chosen here were such that (2.12a) holds: technically once σ reaches a value of $O(Re^{-\frac{1}{38}})$ then the shock layer problem becomes nonlinear itself for we can no longer ignore viscous effects in entropy and shear waves which are produced when the acoustic wave meets the shock. Finally, we mention the observation of CH who concluded that in addition to the neutral modes sketched in Figure (2) there are other modes with zero growth rate but with infinite frequency and therefore these modes cannot be described within the asymptotic framework used here. Investigation of both the linearised and nonlinear problems which arise within the necessarily modified structure would be extremely worthwhile.

There are, therefore, many directions in which the results presented here may be extended. Progress in any of these would be valuable indeed.

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APPENDIX

In this appendix we summarise the results of manipulating (4.28) and using the first and second order solutions in order to derive the desired amplitude equation. We recall that $\alpha \hat{U}_1 + \beta_1 \hat{W}_1$ is given by (4.8) and by considering the $O(h)$ terms in the lower deck equation (3.3d) we obtain $\hat{W}_1$. Similarly, W_{21} can be found by examining $O(h^2 E^2)$ terms in this equation and then U_{21} derived via (4.15a). Finally, $\hat{V}_1$ and V_{21} are given by (4.10) and (4.15b) respectively and thus $G(\xi, \tilde{X})$ deduced from its definition (4.24b). We may now proceed to substitute in (4.28) and the resulting classical Stuart–Watson type evolution equation for $\hat{A}_1(\tilde{X})$ is as follows:

If $\beta_1^2 > \alpha^2$ then

$$
\left[\alpha^{\frac{2}{3}} L_1 - \frac{\beta_1^2 \alpha^3 Ai(\xi_0)}{(\beta_1^2 - \alpha^2)^{\frac{3}{2}}} \left(\frac{(2\beta_1^2 - \alpha^2)}{\alpha^2} \tanh(\sqrt{\beta_1^2 - \alpha^2}\bar{y}_s) - \frac{\sqrt{\beta_1^2 - \alpha^2}\bar{y}_s}{\cosh^2(\sqrt{\beta_1^2 - \alpha^2}\bar{y}_s)} \right) \right] \frac{d\hat{A}_1}{d\tilde{X}}
$$

$$
= \left[\alpha^{\frac{5}{3}} L_2 \lambda_2 + \frac{\alpha^{\frac{5}{3}}\beta_2}{\beta_1} L_3 + \frac{\alpha^2 \beta_1^3 \beta_2}{(\beta_1^2 - \alpha^2)^{\frac{3}{2}}} L_4 \left(\tanh(\sqrt{\beta_1^2 - \alpha^2}\bar{y}_s) - \frac{\sqrt{\beta_1^2 - \alpha^2}\bar{y}_s}{\cosh^2(\sqrt{\beta_1^2 - \alpha^2}\bar{y}_s)} \right) \right] \hat{A}_1
$$

$$
+ \alpha^{\frac{7}{3}} \left(\frac{L_5 + \Phi_+ L_6}{L_7 + \Phi_+ L_8} \right) \hat{A}_1 |\hat{A}_1|^2,
$$

$$(A.1a)$$

where

$$
\Phi_+ \equiv \frac{2\alpha^{\frac{1}{3}}\beta_1^2}{\sqrt{\beta_1^2 - \alpha^2}} \tanh(2\sqrt{\beta_1^2 - \alpha^2}\bar{y}_s),
$$

and where $\{L_j\}_{j=1}^{j=8}$ are complex constants defined presently. If however $\beta_1^2 < \alpha^2$ then

$$
\left[\alpha^{\frac{2}{3}} L_1 + \frac{\beta_1^2 \alpha^3 Ai(\xi_0)}{(\alpha^2 - \beta_1^2)^{\frac{3}{2}}} \left(\frac{(2\beta_1^2 - \alpha^2)}{\alpha^2} \tan(\sqrt{\alpha^2 - \beta_1^2}\bar{y}_s) - \frac{\sqrt{\alpha^2 - \beta_1^2}\bar{y}_s}{\cos^2(\sqrt{\alpha^2 - \beta_1^2}\bar{y}_s)} \right) \right] \frac{d\hat{A}_1}{d\tilde{X}}
$$

$$
= \left[\alpha^{\frac{5}{3}} L_2 \lambda_2 + \frac{\alpha^{\frac{5}{3}}\beta_2}{\beta_1} L_3 - \frac{\alpha^2 \beta_1^3 \beta_2}{(\alpha^2 - \beta_1^2)^{\frac{3}{2}}} L_4 \left(\tan(\sqrt{\alpha^2 - \beta_1^2}\bar{y}_s) - \frac{\sqrt{\alpha^2 - \beta_1^2}\bar{y}_s}{\cos^2(\sqrt{\alpha^2 - \beta_1^2}\bar{y}_s)} \right) \right] \hat{A}_1
$$

$$
+ \alpha^{\frac{7}{3}} \left(\frac{L_5 + \Phi_- L_6}{L_7 + \Phi_- L_8} \right) \hat{A}_1 |\hat{A}_1|^2,
$$

$$(A.1b)$$

where

$$
\Phi_- \equiv \frac{2\alpha^{\frac{1}{3}}\beta_1^2}{\sqrt{\alpha^2 - \beta_1^2}} \tan(2\sqrt{\alpha^2 - \beta_1^2}\bar{y}_s).
$$

The constants here are defined by

$$L_1 = \frac{(T1)}{\kappa} - \frac{i^{-\frac{1}{3}} Ai'(\xi_0)}{\kappa} \int_{\xi_0}^{\infty} K(\xi) Ai(\xi) \left(\int_{\xi_0}^{\xi} \frac{ds}{[Ai(s)]^2} \int_{\infty}^{s} Ai(t) dt \right) d\xi, \qquad (A.2a)$$

$$L_2 = -\frac{i(T1)}{\kappa} - \frac{2 i^{\frac{2}{3}} Ai(\xi_0) Ai'(\xi_0)}{\kappa}, \qquad (A.2b)$$

$$L_3 = \frac{i^{\frac{2}{3}} Ai'(\xi_0)}{\kappa} \left[Ai(\xi_0) - \int_{\xi_0}^{\infty} K(\xi) Ai(\xi) \left(\int_{\xi_0}^{\xi} \frac{ds}{[Ai(s)]^2} \int_{\infty}^{s} Ai(t) dt \right) d\xi \right], \qquad (A.2c)$$

$$L_4 = -i Ai(\xi_0), \qquad (A.2d)$$

$$L_5 = \frac{(T20)(T14) + (T21)(T7)}{\kappa |\kappa|^2}, \qquad (A.2e)$$

$$L_6 = \frac{(T20)(T13) - (T21)(T12)}{\kappa |\kappa|^2}, \qquad (A.2f)$$

$$L_7 = -2^{\frac{1}{3}} \left[\frac{d}{d\hat{\xi}} \left(Ai(\hat{\xi}) \right) \right]_{\hat{\xi} = 2^{\frac{1}{3}} \xi_0}, \qquad (A.2g)$$

and

$$L_8 = 2^{\frac{2}{3}} i^{\frac{1}{3}} \int_{2^{\frac{1}{3}} \xi_0}^{\infty} Ai(s) ds. \qquad (A.2h)$$

The complex constants $T1, T7, T12, T13, T14, T20$ and $T21$ are defined and evaluated in the Appendix of Bassom (1989). All the above expressions were found numerically. Initial conditions at a suitably large value of ξ, say ξ_∞, were found using asymptotic methods and the defining equations integrated using a fourth order Runge–Kutta scheme. The integrations necessary in the evaluation of $(A2)$ were performed using the trapezium rule combined with Richardson extrapolation. Various values of ξ_∞ and step length were used and the results quoted below are believed to be accurate to within 0.1%. Further checks were made on these computed values by using the program to calculate some of the coefficients given by Hall & Smith (1982) and defined in their equation (2.27). Using the above techniques we found that

$$L_1 \approx (0.4766, 0.9858), \quad L_2 \approx (0.0277, -1.202),$$

$$L_3 \approx (1.916, 1.450), \quad L_4 \approx (-0.9573, -0.7247),$$

$$L_5 \approx (-0.8346, 1.389), \quad L_6 \approx (2.092, -0.9825),$$

$$L_7 \approx (-4.633, -0.6513), \quad L_8 \approx (0.3473, 1.536).$$

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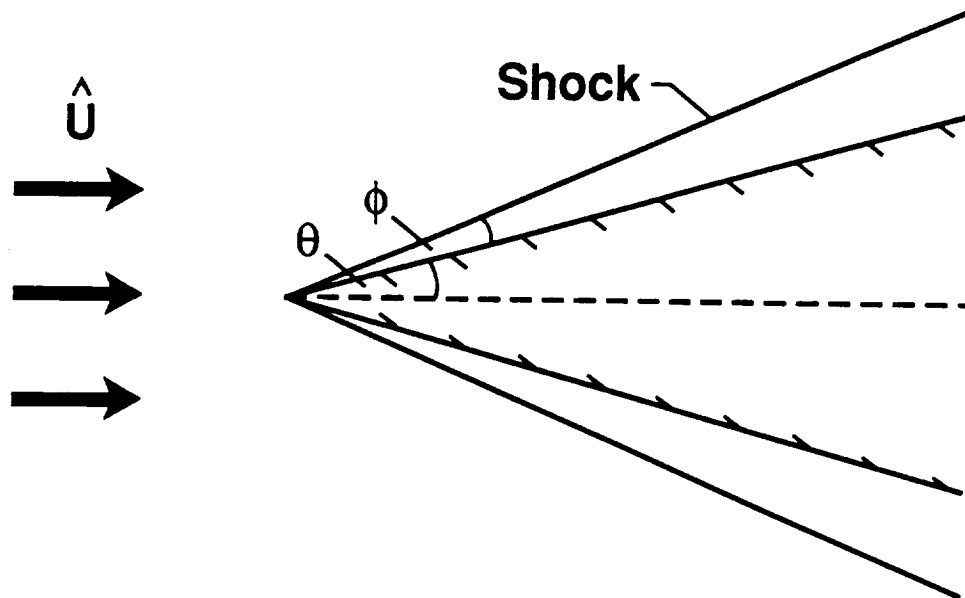
Figure (1). The geometry of the wedge and shock for a high Mach number flow.
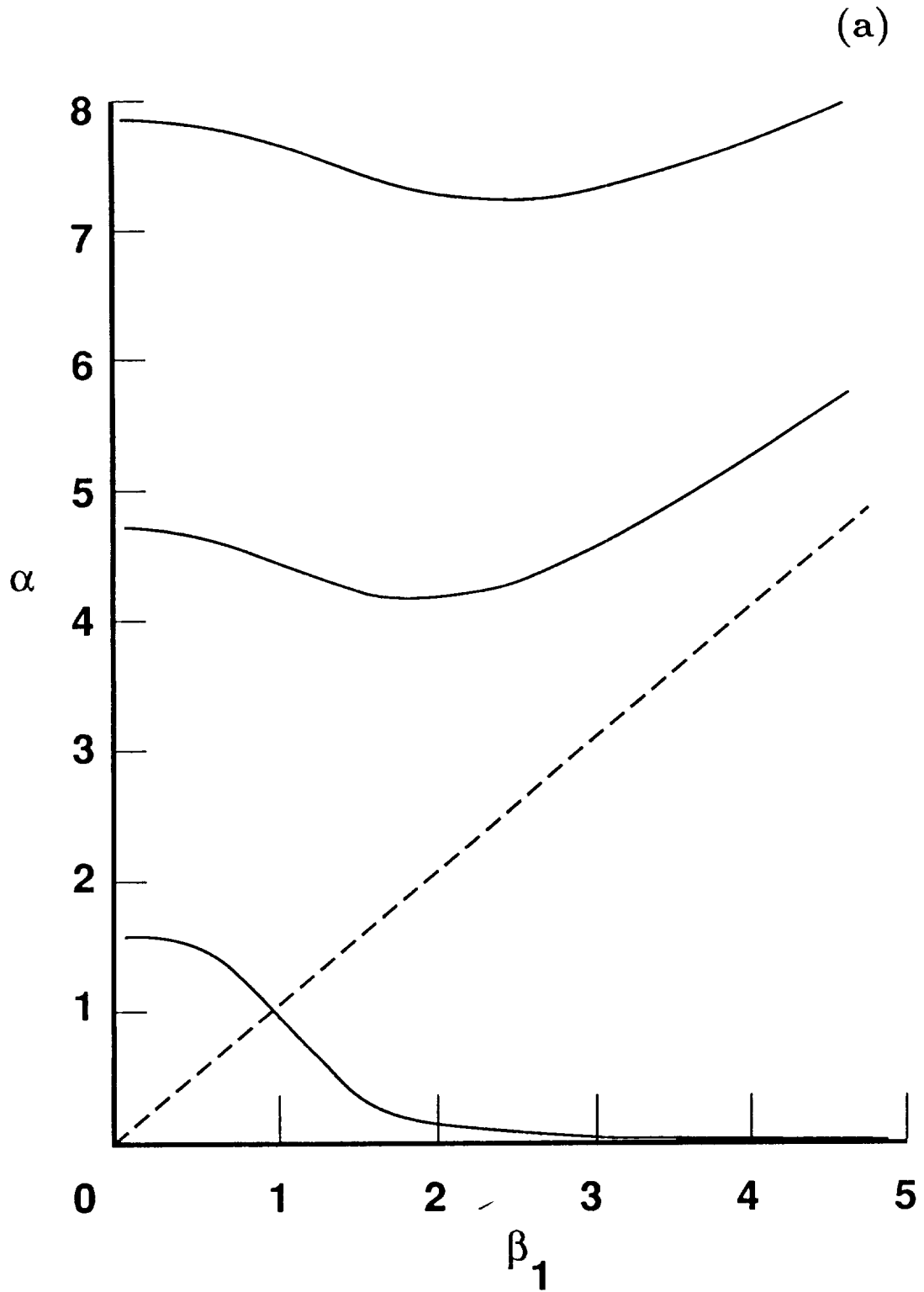
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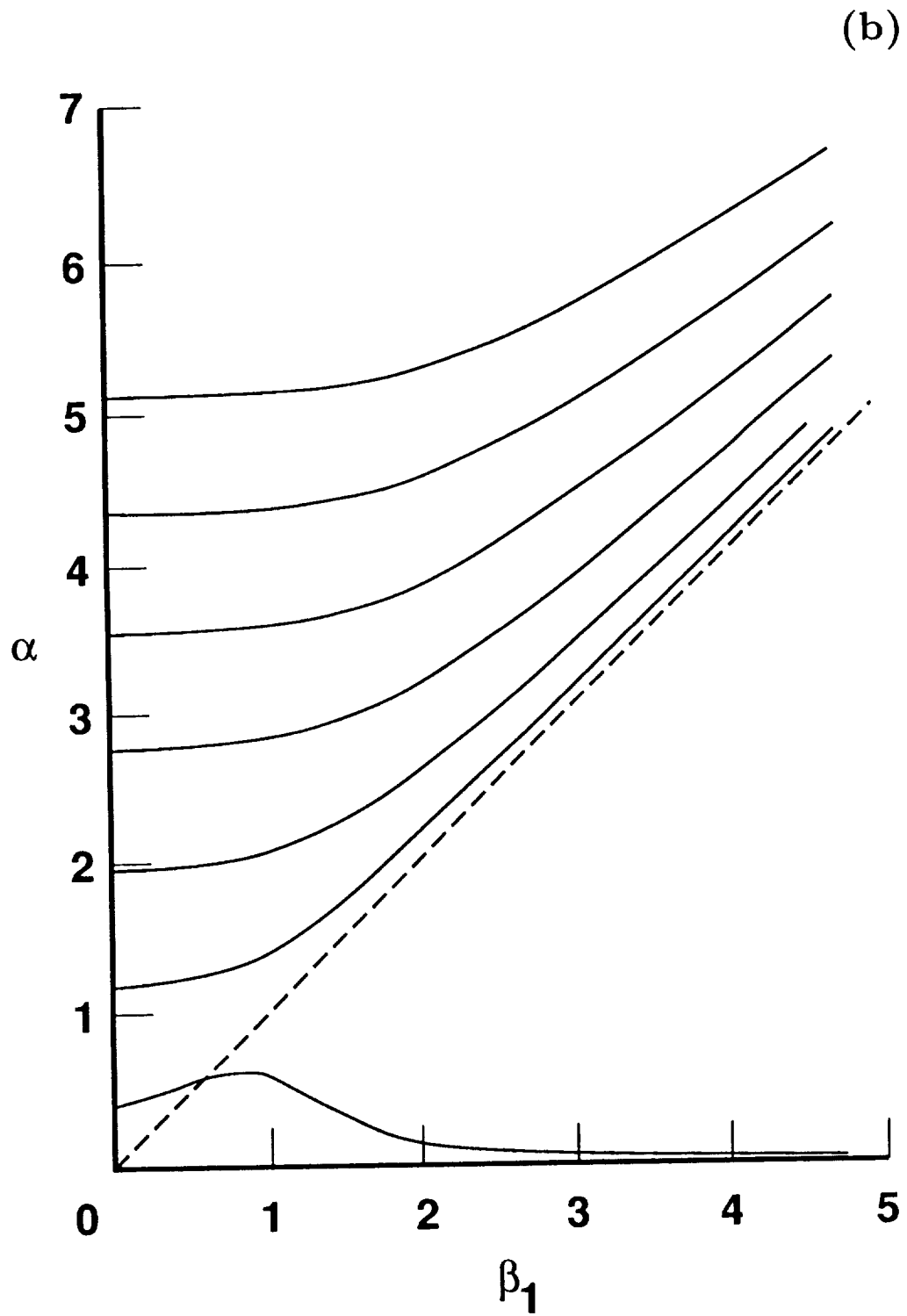
Figure (2). *(a)* The first three neutral curves $\alpha \equiv \alpha(\beta_1)$ given by (4.12) for $\bar{y}_s = 1$. *(b)* The first seven neutral curves for $\bar{y}_s = 4$. *(c)* The first ten neutral curves for $\bar{y}_s = 16$.

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(c)

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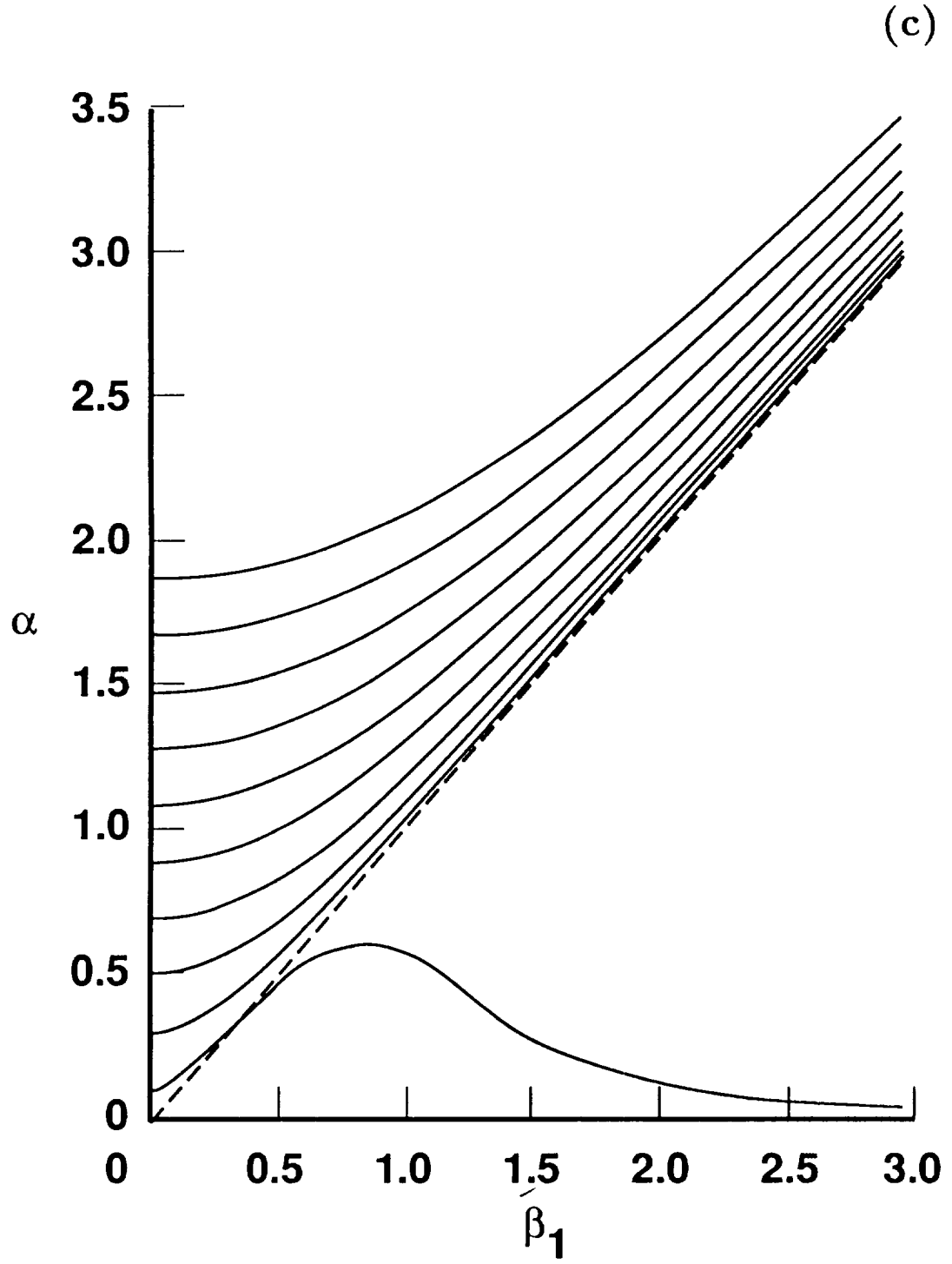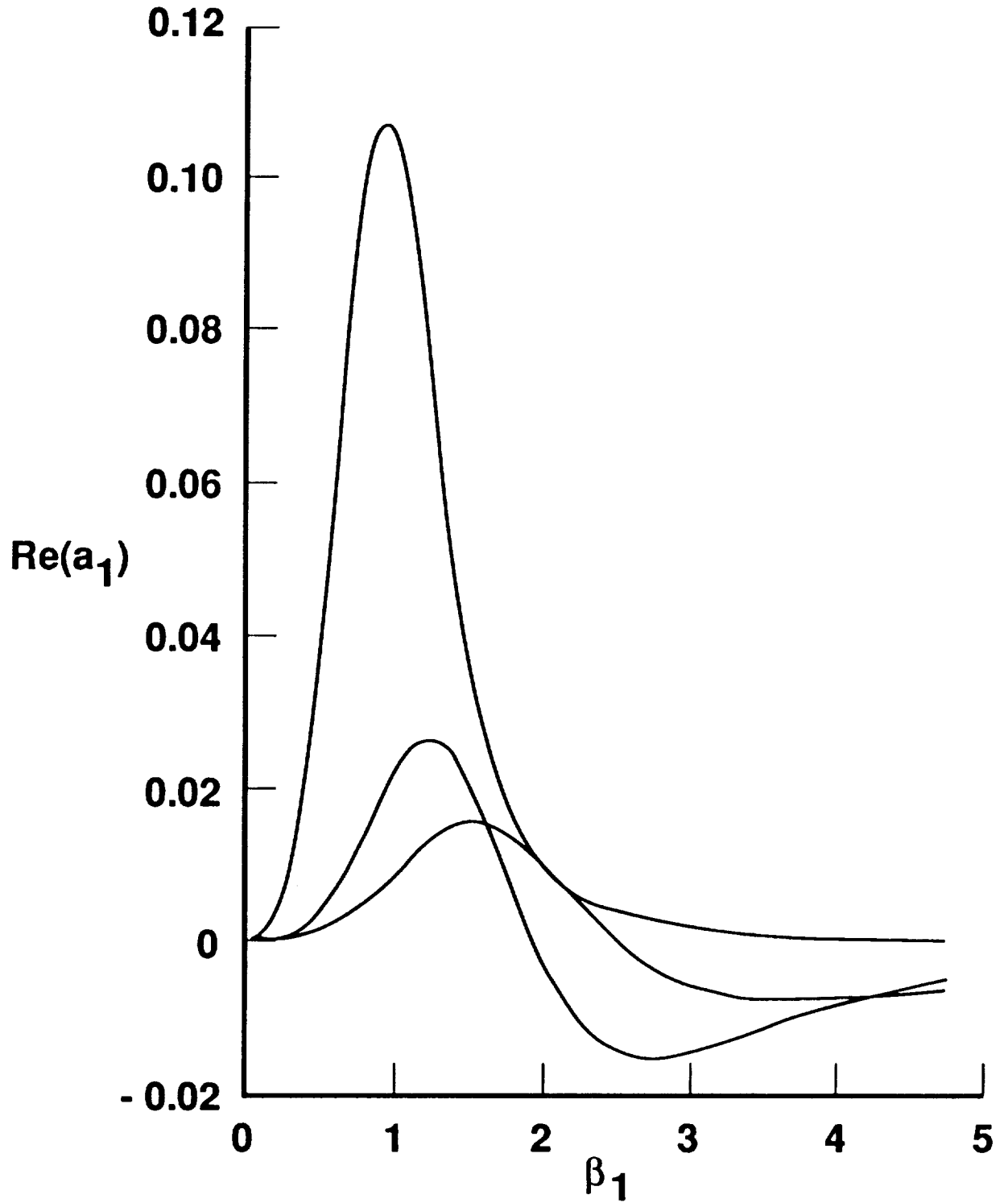
Figure (3). The real part of the coefficient $a_1(\beta_1)$ as defined in (5.1). *(a)* The values taken on the three curves in Figure (2a) where $\bar{y}_s = 1$. *(b)* The values taken on the seven curves in Figure (2b) where $\bar{y}_s = 4$. *(c)* The values taken on the ten curves in Figure (2c) where $\bar{y}_s = 16$.
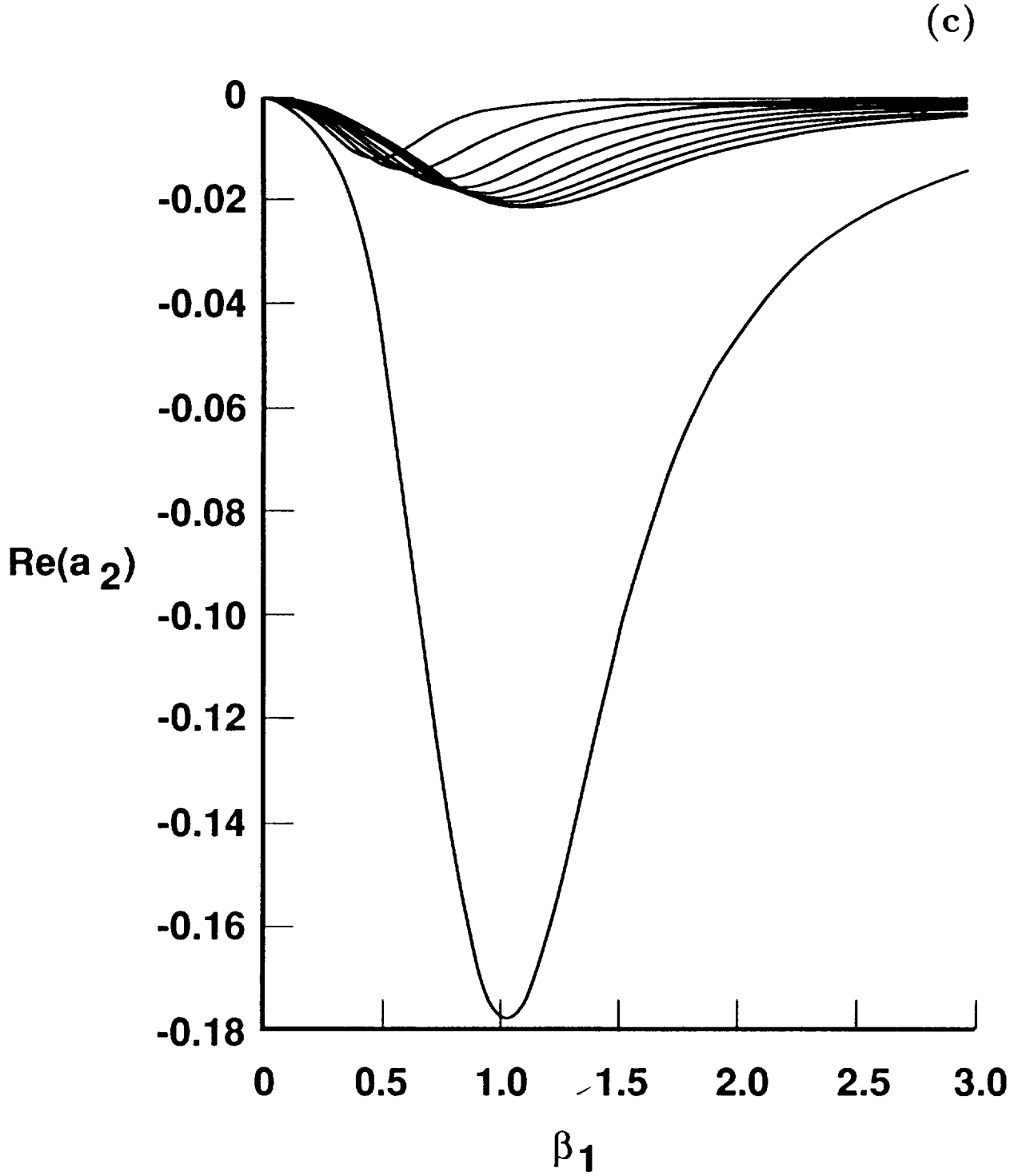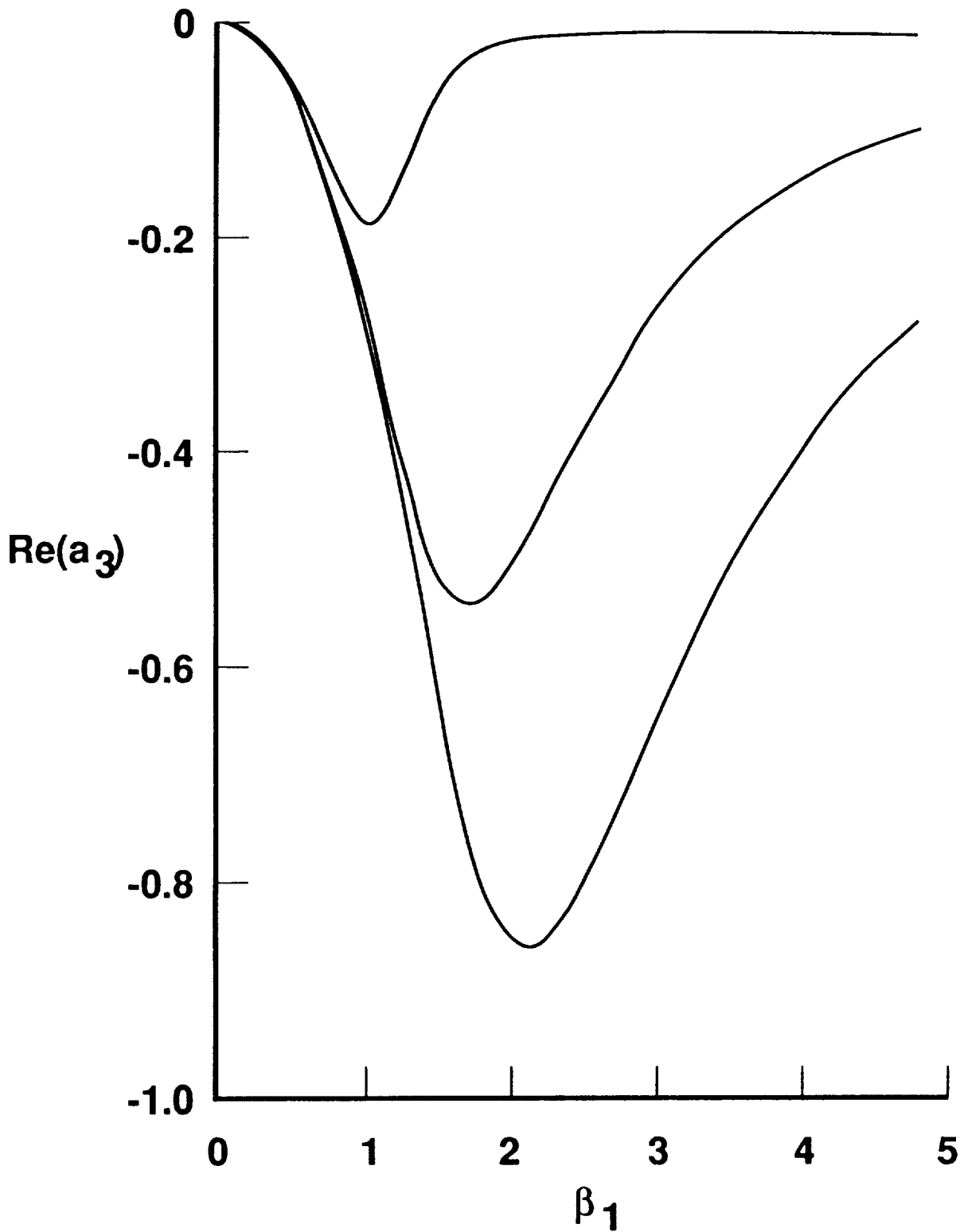
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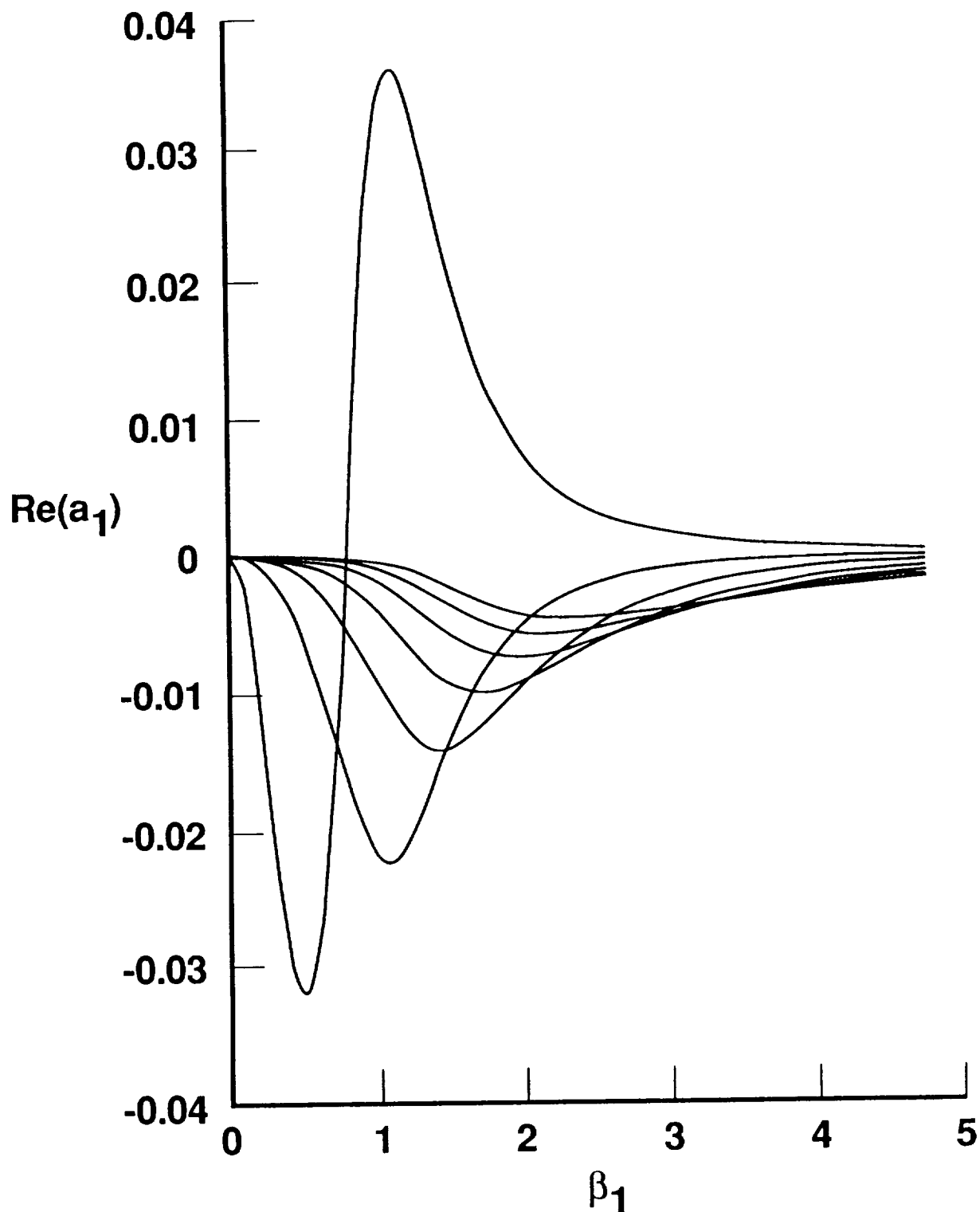
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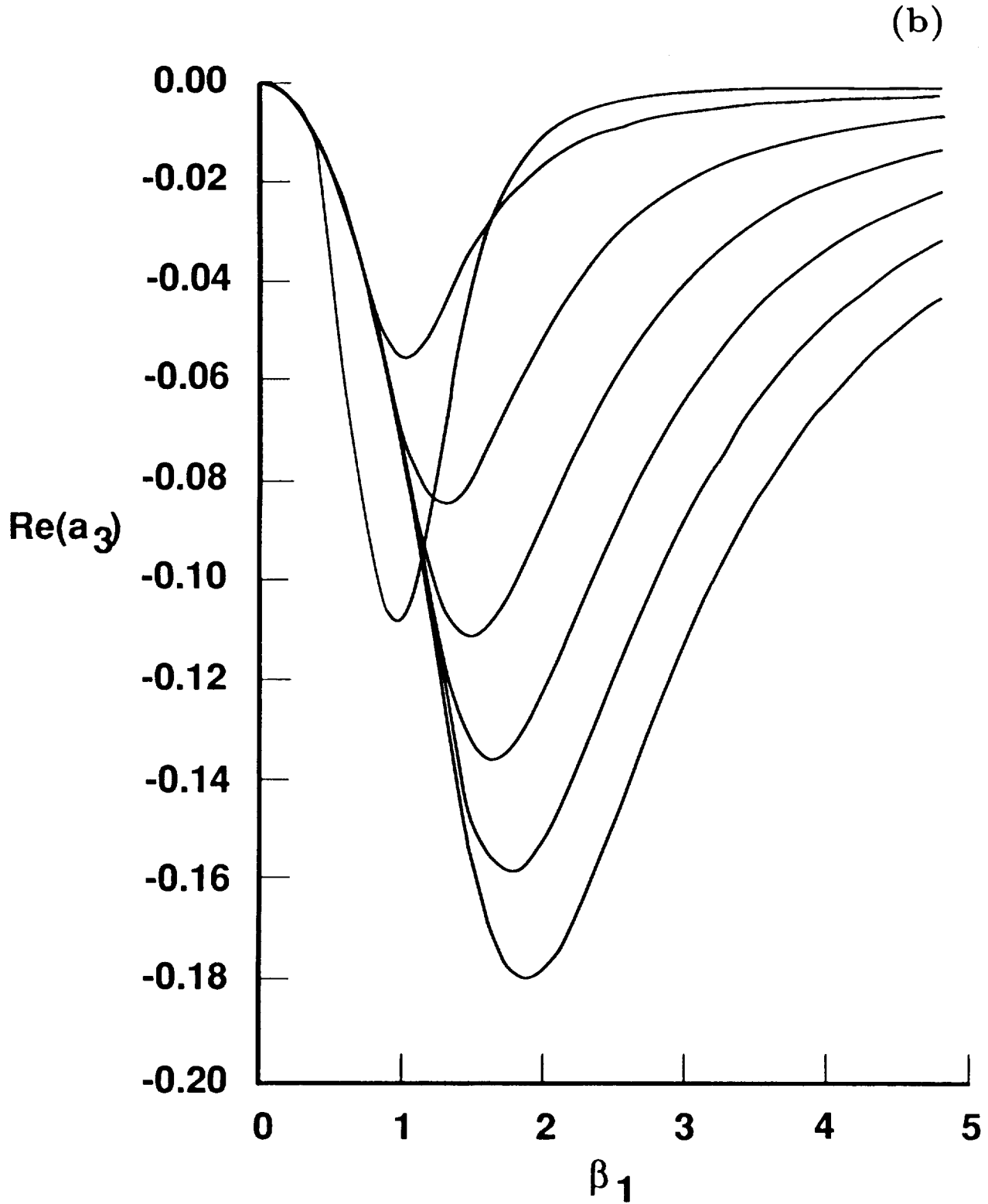
(a)

Figure (4). The real part of the coefficient $a_2(\beta_1)$ as defined in (5.1). *(a)* The values taken on the three curves in Figure (2a) where $\bar{y}_s = 1$. *(b)* The values taken on the seven curves in Figure (2b) where $\bar{y}_s = 4$. *(c)* The values taken on the ten curves in Figure (2c) where $\bar{y}_s = 16$.
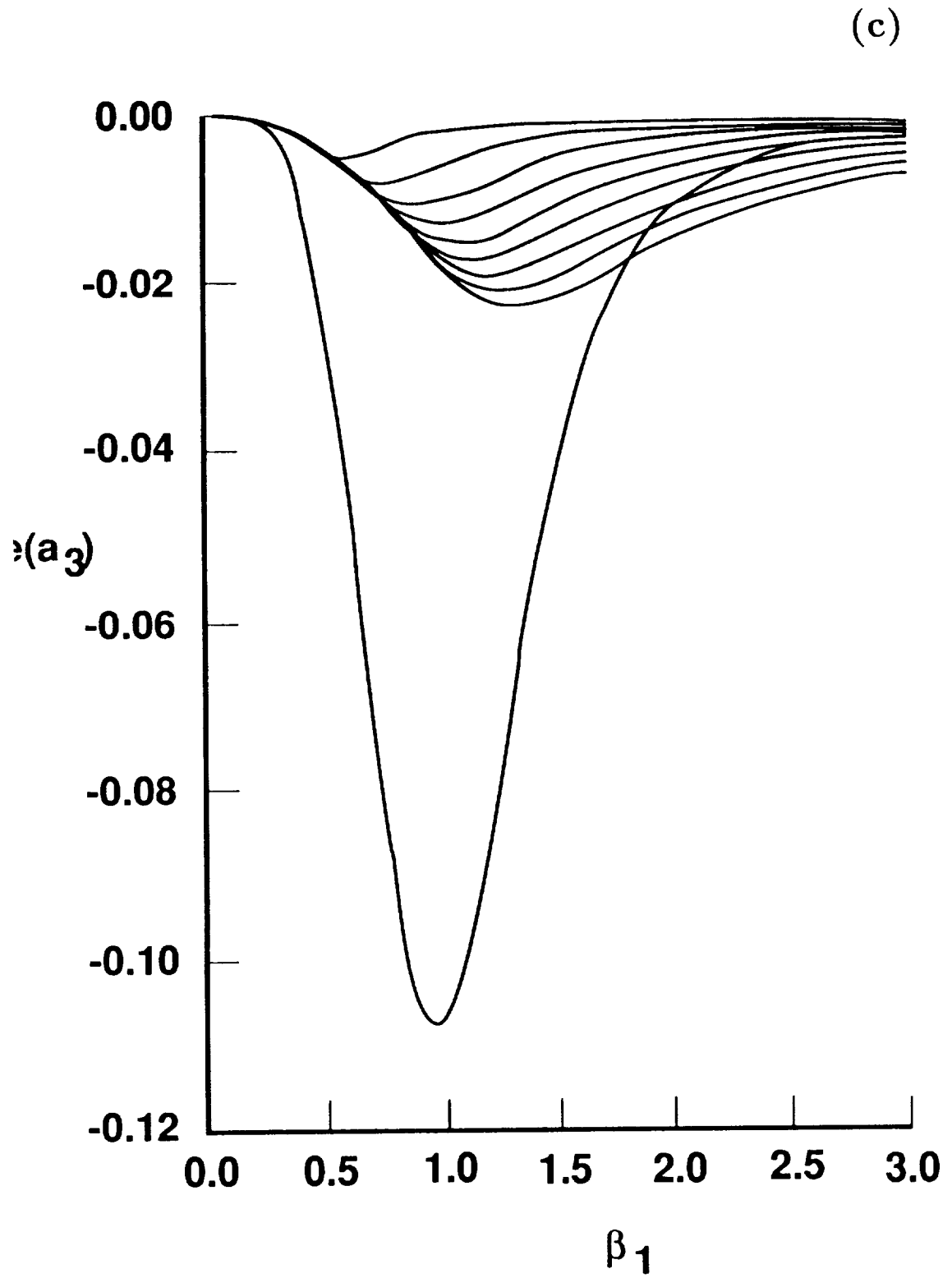
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(b)

Figure (4). The real part of the coefficient $a_2(\beta_1)$ as defined in (5.1). *(a)* The values taken on the three curves in Figure (2a) where $\bar{y}_s = 1$. *(b)* The values taken on the seven curves in Figure (2b) where $\bar{y}_s = 4$. *(c)* The values taken on the ten curves in Figure (2c) where $\bar{y}_s = 16$.

(c)

Figure (4). The real part of the coefficient $a_2(\beta_1)$ as defined in (5.1). (a) The values taken on the three curves in Figure (2a) where $\bar{y}_s = 1$. (b) The values taken on the seven curves in Figure (2b) where $\bar{y}_s = 4$. (c) The values taken on the ten curves in Figure (2c) where $\bar{y}_s = 16$.

Figure (5). The real part of the nonlinear coefficient $a_3(\beta_1)$ as defined in (5.1). *(a)* The values taken on the three curves in Figure *(2a)* where $\bar{y}_s = 1$. *(b)* The values taken on the seven curves in Figure *(2b)* where $\bar{y}_s = 4$. *(c)* The values taken on the ten curves in Figure *(2c)* where $\bar{y}_s = 16$.

Figure (5). The real part of the nonlinear coefficient $a_3(\beta_1)$ as defined in (5.1). *(a)* The values taken on the three curves in Figure *(2a)* where $\bar{y}_s = 1$. *(b)* The values taken on the seven curves in Figure *(2b)* where $\bar{y}_s = 4$. *(c)* The values taken on the ten curves in Figure *(2c)* where $\bar{y}_s = 16$.

Figure (5). The real part of the nonlinear coefficient $a_3(\beta_1)$ as defined in (5.1). *(a)* The values taken on the three curves in Figure (2a) where $\bar{y}_s = 1$. *(b)* The values taken on the seven curves in Figure (2b) where $\bar{y}_s = 4$. *(c)* The values taken on the ten curves in Figure (2c) where $\bar{y}_s = 16$.

16. Abstract

The nonlinear stability of a compressible flow past a wedge is investigated in the hypersonic limit. The analysis follows the ideas of a weakly nonlinear approach as first detailed by Smith (1979). Interest is focussed on Tollmien-Schlichting waves governed by a triple deck structure and it is found that the attached shock can profoundly affect the stability characteristics of the flow. In particular, it is shown that nonlinearity tends to have a stabilising influence. The nonlinear evolution of the Tollmien-Schlichting mode is described in a number of asymptotic limits which were first identified by Cowley and Hall (1990) in their linearised account of the current problem.